1973

Contributions To The Asymptotically Best Linear-estimates Based On Selected Order-statistics

Smiley Wei-hsiao Cheng

CONTRIBUTIONS

TO

THE ASYMPTOTICALLY BEST LINEAR ESTIMATES

BASED ON SELECTED ORDER STATISTICS

by

Cheng, Smiley Wei-Hsiao

Department of Mathematics

Submitted in partial fulfilment

of the requirements for the degree of

Doctor of Philosophy

Faculty of Graduate Studies

The University of Western Ontario

London, Canada

July 1973

ABSTRACT

The linear combinations of order statistics are very useful due to computational simplicity. When some of the observations in a sample are missing, linear estimates based on order statistics are particularly useful since standard methods of estimation tend to become laborious and unsatisfactory.

When sample size is very large, the estimates based on a few suitably chosen order statistics from the sample usually give high efficiency. How to choose these order statistics in order to give the highest efficiency among all the estimates based on the same number of chosen order statistics is a very important technique.

Ogawa first proposed the asymptotically best linear estimate (ABLE) and the determination of the optimum spacing which will give the optimum ranks of order statistics.

In this thesis, the main portion concentrates on the determination of the optimum spacing for the ABLE in the following three different distributions with corresponding parameters which is to be estimated. Estimations are all based on complete and censored samples:

(1) Logistic distribution - location parameter.

(2) Pareto distribution - scale parameter.

(3) Rayleigh distribution - scale parameter.

We also do the same study for the case of multiply censored sample.

Finally, we consider an asymptotically uniformly most powerful test of the location parameter using the ABLE.

ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to his research advisor, Dr. L.K. Chan for suggesting the problems, his keen interest in the work and his constant advice, guidance and encouragement.

The author would like to thank Dr. E.R. Mead for reading this thesis carefully and for his valuable suggestion which improves this thesis considerably.

The author wants to thank National Research Council of Canada, the Department of University Affairs, Ontario, and the Department of Mathematics of the University of Western Ontario for their financial assistance.

Finally, the author is indebted to Mrs. C. Cloutier for her skillful typing and to his wife for her understanding, patience and encouragement.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

Consider a sample of n order statistics

$$X_{(1)} < X_{(2)} < \cdots < X_{(n)}$$

from a continuous distribution with probability density function $(1/\sigma)f[(x-\mu)/\sigma]$, where μ and σ are location and scale parameters respectively.

The use of linear combinations of order statistics for the estimation of μ and/or σ is very popular due to the computational simplicity, provided the coefficients of the order statistics are available. Furthermore, even when the sample size is very large, estimates based on a few suitably chosen order statistics from the sample usually still give high efficiency.

In situations where some of the observations in a sample are missing, linear estimates based on order statistics are particularly useful since standard methods of estimation tend to become laborious or otherwise unsatisfactory. Typical examples occur in life testing where an experiment is terminated after the first r out of n items under test have failed.

A sample in which part of the observations are not available is called a censored sample. The censored samples we consider are the following:

(1) Singly censored sample - either the values of the r_1 smallest (called left censoring) or the r_2 largest (called right censoring) observations are not available, where r_1 or r_2 is a fixed number.

(2) Doubly censored sample - the values of the r_1 smallest and r_2 largest observations are not available, where r_1 and r_2 are both fixed numbers.

(3) Multiply censored sample - if an ordered sample is partitioned into m (a fixed number) groups of observations in which the alternative groups are missing, the sample is defined as a multiply censored sample.

It is easily seen that complete samples, singly censored samples and doubly censored samples are special cases of multiply censored samples.

In the estimation of a parameter by a linear combination of k(<n) chosen order statistics from a censored sample, the set of k order statistics which gives the minimum

variance among all possible choices of k order statistics is the preferable one. The problem is then to determine the ranks n_i, $i = 1, 2, \ldots, k$ of such optimum order statistics (called optimum ranks). When n is small, one usually can compare the values of all the variances of the linear estimates. However, when n or k is large, this procedure is prohibitively time-consuming, even with the use of a computer. Asymptotic theory is then useful for the approximation of such optimum ranks.

Given a set of k fixed values $\{\lambda_i\} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < 1$ which is called a spacing, define $n_i = [n\lambda_i]+1$, where $[n\lambda_i]$ represents the greatest integer not exceeding $n\lambda_i$. The problem is to find a suitable spacing which minimizes the asymptotic variance of the linear estimate whose ranks are determined by $\{\lambda_i\}$ (called an optimum spacing.)

Ogawa (1951) first considered the above problems and gave a systematical study. He proposed the asymptotically best linear estimate (ABLE) and proceeded to the determination of the optimum spacings for the parameters of the normal distribution. His work has stimulated a number of further studies:

Normal distribution - Ogawa (1951, 1972), Kulldorff (1958, 1963, 1964).

Exponential distribution - Ogawa (1960), Kulldorff (1962, 1963), Saleh and Ali (1966).

Cauchy distribution - Chan (1970), Balmer, Boulton and Sack (1972).

Logistic distribution - Chan (1969), Gupta and Gnanadesikan (1966), Chan and Cheng (1972).

Extre.-value distribution - Chan and Kabir (1969), Hassanein (1968).

Pareto distribution - Kulldorff and Vännman (1973).

The main portion of this thesis concentrates on the determination of the optimum spacings for the ABLE of the location parameter of the logistic distribution, the scale parameter of the Pareto and Rayleigh distributions based on censored samples and the development of an algorithm to find an optimum spacing for the ABLE based on a multiply censored sample in which we present a verification of the algorithm for the ABLE of the location parameter of the logistic distribution and give a discussion of the application to the other distributions.

An asymptotically uniformly most powerful test (in the sense that an uniformly most powerful test based on the limiting distribution of the order statistics) of the location parameter using the ABLE is also proposed. The test is a one sided test.

In Chapter 2, we give a brief summary of the theory of the ABLE.

Exponential distribution - Ogawa (1960), Kulldorff (1962, 1963), Saleh and Ali (1966).

Cauchy distribution - Chan (1970), Balmer, Boulton and Sack (1972).

Logistic distribution - Chan (1969), Gupta and Gnanadesikan (1966), Chan and Cheng (1972).

Extre.-value distribution - Chan and Kabir (1969), Hassanein (1968).

Pareto distribution - Kulldorff and Vännman (1973).

The main portion of this thesis concentrates on the determination of the optimum spacings for the ABLE of the location parameter of the logistic distribution, the scale parameter of the Pareto and Rayleigh distributions based on censored samples and the development of an algorithm to find an optimum spacing for the ABLE based on a multiply censored sample in which we present a verification of the algorithm for the ABLE of the location parameter of the logistic distribution and give a discussion of the application to the other distributions.

An asymptotically uniformly most powerful test (in the sense that an uniformly most powerful test based on the limiting distribution of the order statistics) of the location parameter using the ABLE is also proposed. The test is a one sided test.

In Chapter 2, we give a brief summary of the theory of the ABLE.

Exponential distribution - Ogawa (1960), Kulldorff (1962, 1963), Saleh and Ali (1966).

Cauchy distribution - Chan (1970), Balmer, Boulton and Sack (1972).

Logistic distribution - Chan (1969), Gupta and Gnanadesikan (1966), Chan and Cheng (1972).

Extre.-value distribution - Chan and Kabir (1969), Hassanein (1968).

Pareto distribution - Kulldorff and Vännman (1973).

The main portion of this thesis concentrates on the determination of the optimum spacings for the ABLE of the location parameter of the logistic distribution, the scale parameter of the Pareto and Rayleigh distributions based on censored samples and the development of an algorithm to find an optimum spacing for the ABLE based on a multiply censored sample in which we present a verification of the algorithm for the ABLE of the location parameter of the logistic distribution and give a discussion of the application to the other distributions.

An asymptotically uniformly most powerful test (in the sense that an uniformly most powerful test based on the limiting distribution of the order statistics) of the location parameter using the ABLE is also proposed. The test is a one sided test.

In Chapter 2, we give a brief summary of the theory of the ABLE.

In Chapter 3, an unified method is developed for determining the optimum spacing of the ABLE of the parameter based on complete, singly censored and doubly censored samples, when the given distribution satisfy certain conditions. The method is applied to the logistic, Pareto and Rayleigh distributions in Chapters 4, 5 and 6 respectively, when the sample is complete, singly or doubly censored.

An algorithm used for determining the optimum spacing when the sample is multiply censored is given in Chapter 7.

Chapter 8 gives an asymptotically uniformly most powerful (UMP) test of the location parameter using the ABLE.

Unified method can also be applied to the following cases for finding the optimum spacing of the ABLE when complete, singly censored and doubly censored samples are available

(1) extreme-value distribution - location parameter

(2) exponential distribution - scale parameter

(3) Weibull distribution - scale parameter.

These problems have been solved by various authors using different approaches such as Ali and Saleh (1966) and Chan and Kabir (1969). Thus we will not consider these cases.

CHAPTER 2

THE ASYMPTOTICALLY BEST LINEAR ESTIMATES OF THE PARAMETERS BASED ON k ORDER STATISTICS

2.1 The ABLE's

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with probability density function (pdf) $(1/\sigma)f[(x-\mu)/\sigma]$ and cumulative distribution function (cdf) $F[(x-\mu)/\sigma]$. Rearranging them in the order of their magnitudes, we call

$$X_{(1)} < X_{(2)} < \ldots < X_{(n)} \tag{2.1}$$

the order statistics. Corresponding to the sample μ and σ are called the location and scale parameters, respectively.

For any given number λ such that $0 < \lambda < 1$, we define the λ-quantile of the population as the value y_λ such that

$$\int_{-\infty}^{y_\lambda} f(y)dy = \lambda \ .$$

A set of $k \leq n$ real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ satisfying the relation

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < 1$$

is called a spacing (a spacing is sometimes called a point and denoted by $(\lambda_1, \lambda_2, \ldots, \lambda_k)$). We call the k order statistics (from (2.1))

$$X_{(n_1)} < X_{(n_2)} < \ldots < X_{(n_k)} \tag{2.2}$$

where $n_i = [n\lambda_i]+1$ and $[n\lambda_i]$ represents the greatest integer not exceeding $n\lambda_i$, the sample quantiles.

Mosteller (1946) proved that the pdf of the limiting distribution of the k sample quantiles (2.2) is

$$\ell(x_{(n_1)}, \ldots, x_{(n_k)}) = h(\lambda_1, \ldots, \lambda_k, \sigma, n) \exp\{-\frac{n}{2\sigma^2}\Omega\},$$

where

$$h(\lambda_1, \ldots, \lambda_k, \sigma, n) = \left(\frac{n}{2\pi\sigma^2}\right)^{k/2} f_1 \ldots f_k \{\lambda_1(\lambda_2 - \lambda_1) \cdot$$

$$\ldots(\lambda_k - \lambda_{k-1})(1-\lambda_k)\}^{-1/2},$$

$$\Omega = \sum_{i=1}^{k} \frac{\lambda_{i+1} - \lambda_{i-1}}{(\lambda_{i+1}-\lambda_i)(\lambda_i-\lambda_{i-1})} f_i^2 (x_{(n_i)} - \mu - \sigma u_i)^2$$

$$- 2 \sum_{i=2}^{k} \frac{f_i f_{i-1}}{\lambda_i - \lambda_{i-1}} (x_{(n_i)} - \mu - \sigma u_i)(x_{(n_{i-1})} - \mu - \sigma u_{i-1}),$$

$$u_i = F^{-1}(\lambda_i), \quad f_i = f(u_i), \quad i = 1, 2, \ldots, k,$$

$$\lambda_o = 0, \quad \lambda_{k+1} = 1.$$

When the sample size is large, Ogawa (1951) obtained the following asymptotically best linear estimate (ABLE) of μ or σ when the other is known by the method of maximum likelihood. ("best" is in the sense that the estimate minimizes asymptotic variances among all the asymptotically unbiased linear estimates based on the same sample quantiles).

Case (i): σ is known. The ABLE of the location parameter μ is

$$\mu^* = \sum_{i=1}^{k} a_i x_{(n_i)} - \sigma \frac{K_3}{K_1} ,$$

where

$$K_1 = \sum_{i=1}^{k+1} \frac{(f_i - f_{i-1})^2}{\lambda_i - \lambda_{i-1}} ,$$

$$K_3 = \sum_{i=1}^{k+1} \frac{(f_i - f_{i-1})(f_i u_i - f_{i-1} u_{i-1})}{\lambda_i - \lambda_{i-1}} ,$$

$$a_i = \frac{f_i}{K_1} \left\{ \frac{f_i - f_{i-1}}{\lambda_i - \lambda_{i-1}} - \frac{f_{i+1} - f_i}{\lambda_{i+1} - \lambda_i} \right\} , \quad i = 1, 2, \ldots, k,$$

$$\lambda_o = 0, \ \lambda_{k+1} = 1, \ \ f_o = f_{k+1} = f_o u_o = f_{k+1} u_{k+1} = 0 .$$

The asymptotic variance of μ^* is

$$A\ V\ (\mu^*) = \frac{\sigma^2}{n} \cdot \frac{1}{K_1} .$$

Case (ii): μ is known. The ABLE of the scale parameter σ is

$$\sigma^* = \sum_{i=1}^{k} b_i x_{(n_i)} - \mu \frac{K_3}{K_2} ,$$

where

$$K_2 = \sum_{i=1}^{k+1} \frac{(f_i u_i - f_{i-1} u_{i-1})^2}{\lambda_i - \lambda_{i-1}} ,$$

$$b_i = \frac{f_i}{K_2} \left(\frac{f_i u_i - f_{i-1} u_{i-1}}{\lambda_i - \lambda_{i-1}} - \frac{f_{i+1} u_{i+1} - f_i u_i}{\lambda_{i+1} - \lambda_i} \right) ,$$

$i = 1, 2, \ldots, k.$

The asymptotic variance of σ^* is

$$A\,V\,(\sigma^*) = \frac{\sigma^2}{n} \cdot \frac{1}{K_2} .$$

Asymptotic relative efficiencies (ARE's) are relative to the Cramér-Rao lower bounds based on all order statistics in (2.1).

The set of values $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ which maximizes K_1 and K_2 is called the optimum spacing for the ABLE μ^* and σ^* respectively.

2.2 The Samples

Definition 2.1: (2.1) is called a complete sample.

Definition 2.2: If only the order statistics

$$X_{(1)} < X_{(2)} < \ldots < X_{([n\beta]+1)}, \quad 0 < \beta < 1, \qquad (2.3)$$

are available while the largest $100(1-\beta)\%$ of the order statistics in (2.1) are missing, the sample is called a right censored sample.

Definition 2.3: If only the order statistics

$$X_{([n\alpha]+1)} < X_{([n\alpha]+2)} < \cdots < X_{(n)}, \quad 0 < \alpha < 1, \qquad (2.4)$$

are available while the smallest $100\alpha\%$ of the order statistics in (2.1) are missing, the sample is called a left censored sample.

Either of the above two cases is called a singly censored sample.

Definition 2.4: If only the order statistics

$$X_{([n\alpha]+1)} < X_{([n\alpha]+2)} < \cdots < X_{([n\beta]+1)}, \quad 0<\alpha<\beta<1, \qquad (2.5)$$

are available while the smallest $[n\alpha]$ and the largest $n-[n\beta]-1$ order statistics are missing, the sample is called a doubly censored sample.

Definition 2.5: If only the order statistics

$$X_{m_1} < X_{m_1+1} < \cdots < X_{m_1+\ell_1} < X_{m_2} < X_{m_2+1} < \cdots <$$

$$X_{m_2+\ell_2} < \cdots < X_{m_J} < X_{m_J+1} < \cdots < X_{m_J+\ell_J} \qquad (2.6)$$

where $m_i = [n\alpha_{i1}]+1$, $m_i+\ell_i = [n\alpha_{i2}]+1$, $i = 1, 2, \ldots, J$, $0 \leq \alpha_{11}<\alpha_{12}<\alpha_{21}<\alpha_{22} < \cdots < \alpha_{J1} < \alpha_{J2} \leq 1$, of (2.1) are available while the other observations are missing, the sample is called a multiply censored sample.

We can easily see that complete, singly censored and doubly censored samples are all special cases of multiply censored sample.

Since an ABLE is based on order statistics selected from the available order statistics in a complete or censored sample which are determined by spacings, it is more convenient to represent complete and censored samples by spacings. So

the complete sample (2.1)

$$\iff \underset{\sim}{\lambda} \varepsilon I(0,1) = \{(\lambda_1,\ldots,\lambda_k) | 0<\lambda_1<\ldots<\lambda_k<1\} \quad ;$$

the right censored sample (2.3)

$$\iff \underset{\sim}{\lambda} \varepsilon I(0,\beta) = \{(\lambda_1,\ldots,\lambda_k) | 0<\lambda_1<\ldots<\lambda_k \leq \beta<1\} \quad ;$$

the left censored sample (2.4)

$$\iff \underset{\sim}{\lambda} \varepsilon I(\alpha,1) = \{(\lambda_1,\ldots,\lambda_k) | 0<\alpha \leq \lambda_1<\ldots<\lambda_k<1\} \quad ;$$

the doubly censored sample (2.5)

$$\iff \underset{\sim}{\lambda} \varepsilon I(\alpha,\beta) = \{(\lambda_1,\ldots,\lambda_k) | 0<\alpha \leq \lambda_1<\ldots<\lambda_k \leq \beta<1\} \quad ;$$

the multiply censored sample (2.6)

$$\iff \underset{\sim}{\lambda} \varepsilon I(\underset{\sim}{\alpha}_1,\underset{\sim}{\alpha}_2) = \{(\lambda_1,\ldots,\lambda_k) | \lambda_i \varepsilon [\alpha_{11},\alpha_{12}] \cup$$

$[\alpha_{21},\alpha_{22}] \cup \ldots \cup [\alpha_{J1},\alpha_{J2}]$ and $\lambda_1<\lambda_2<\ldots<\lambda_k\}$, $0 \leq \alpha_{11}<\alpha_{12}<\alpha_{21}<$ $\alpha_{22}<\ldots<\alpha_{J1}<\alpha_{J2} \leq 1$, where $\underset{\sim}{\lambda} = (\lambda_1,\lambda_2, \ldots, \lambda_k)$.

We also define

$$\overline{I}(a,b) = \{(\lambda_1,\ldots,\lambda_k) \mid 0 \le a \le \lambda_1 \le \cdots \le \lambda_k \le b \le 1\} \quad .$$

CHAPTER 3

THE DETERMINATION OF THE OPTIMUM SPACING FOR THE ABLE θ^* OF A SINGLE PARAMETER θ

3.1 Introduction

We will develop a unified method which leads to the finding of the optimum spacing for the ABLE θ^* of the single parameter θ when the other is known for those distributions which satisfy the conditions in the unified method.

Most of the lemmas and proofs were done by Chan and Kabir (1969) through their work for the Extreme-value distribution. We unify them in this chapter.

3.2 Lemmas and Theorems

In order to determine the optimum spacing for the ABLE θ^* of the parameter θ, we have to minimize the asymptotic variance $A\ V\ (\theta^*) = \sigma^2/[nK(\lambda_1,\ldots,\lambda_k)]$ with respect to $\lambda_1,\lambda_2,\ \ldots,\ \lambda_k$, $K = K_1$ when $\theta^* = \mu^*$ and $K = K_2$ when $\theta^* = \sigma^*$.

Lemma 3.1: The function

$$K = \sum_{i=1}^{k+1} \frac{(m_i - m_{i-1})^2}{\lambda_i - \lambda_{i-1}} \qquad (3.1)$$

where $m_i = m(\lambda_i)$, $i = 1, 2, \ldots, k$, $m_o = m_{k+1} = 0$, is a continuous function of λ_i and has a nonvanishing 2nd derivative, defined on $\overline{I}(0,1)$ attains its maximum at an interior point of $\overline{I}(0,1)$.

Proof: By assumption, m_i is a continuous function of λ_i, $(m_i-m_{i-1})^2/(\lambda_i-\lambda_{i-1}) \to 0$ as $|\lambda_i - \lambda_{i-1}| \to 0$. K is thus continuous on a compact set $\overline{I}(0,1)$. So K attains its maximum at some point, say, $(\lambda_1,\lambda_2,\ldots,\lambda_k)$ in $\overline{I}(0,1)$.

The assumption that m_i has a nonvanishing 2nd derivative gives the fact that $m(\delta)$ is either concave upward or concave downward. Hence for $0\le\delta_1<\delta_2<\delta_3\le 1$, we have

$$\frac{m(\delta_3)-m(\delta_2)}{\delta_3 - \delta_2} \neq \frac{m(\delta_2) - m(\delta_1)}{\delta_2 - \delta_1} \quad .$$

Next, we want to show that the value of K decreases if $\lambda_i = \lambda_{i+1}$ for any $i = 0, 1, 2, \ldots, k$.

Suppose $\lambda_{i-1} = \lambda_i<\lambda_{i+1}$ for some $i = 1,2,\ldots,k$, then for any λ such that $\lambda_{i-1} < \lambda < \lambda_{i+1}$,

$$K(\lambda_1,\ldots,\lambda_{i-1},\lambda_i,\lambda_{i+1},\ldots,\lambda_k)-K(\lambda_1,\ldots,\lambda_{i-1},\lambda,\lambda_{i+1},\ldots,\lambda_k)$$

$$= \frac{(m_{i+1}-m_{i-1})^2}{\lambda_{i+1}-\lambda_{i-1}} - \frac{(m_{i+1}-m)^2}{\lambda_{i+1}-\lambda} - \frac{(m-m_{i-1})^2}{\lambda-\lambda_{i-1}}$$

$$= -\ (\lambda_{i+1}-\lambda_{i-1})\left(\frac{\lambda_{i+1}-\lambda}{\lambda_{i+1}-\lambda_{i-1}}\right)\left(\frac{\lambda-\lambda_{i-1}}{\lambda_{i+1}-\lambda_{i-1}}\right)\left(\frac{m_{i+1}-m}{\lambda_{i+1}-\lambda} - \frac{m-m_{i-1}}{\lambda-\lambda_{i-1}}\right)^2 < 0,$$

since every factor in the product on the right hand side is positive. So the maximum must occur at an interior point.

Hence for maximizing K, it is sufficient to focus on the set $I(0,1)$.

Lemma 3.2: Let $H_1(y)$ and $H_2(y)$ be two monotonic decreasing functions of y, $y\varepsilon[0,1]$. Then for any fixed $y'\varepsilon[0,1)$, there exists a unique $y^*\varepsilon(y',1)$ such that $H_2(y^*) = H_1(y')$ if

(a) $H_2(y) > H_1(y)$, for all $y\varepsilon[0,1)$,

(b) $H_1(1) = H_2(1) =$ constant.

Proof: This lemma can be proved by elementary calculus (Fig. 1).

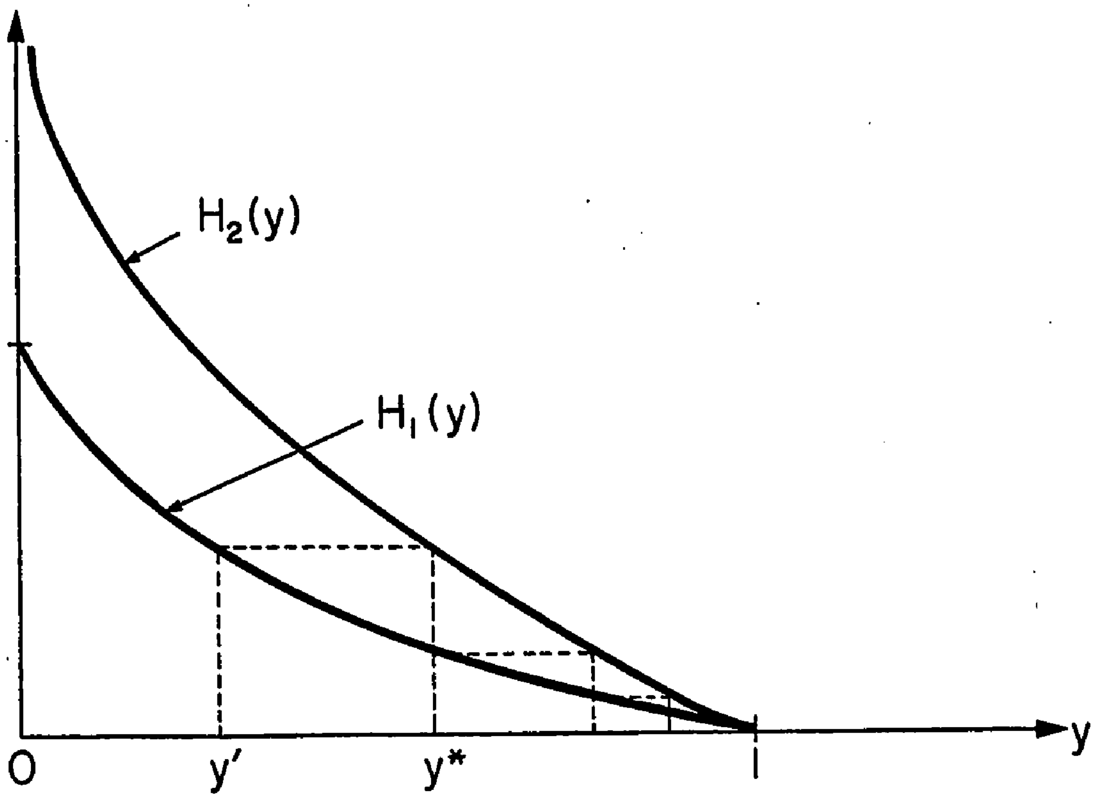

FIG. 1

(1) Complete Sample

If we are given a complete sample (2.1), in order to determine the optimum spacing, we have to maximize K defined on I(0,1).

Lemma 3.3: A point $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ which maximizes the K of (3.1) defined on I(0,1) must satisfy the system of equations

$$M(\lambda_{i-1}, \lambda_i, \lambda_{i+1}) = \frac{m_{i+1}-m_i}{\lambda_{i+1}-\lambda_i} + \frac{m_i-m_{i-1}}{\lambda_i-\lambda_{i-1}} - 2\,\frac{dm_i}{d\lambda_i} = 0 \ , \quad (3.2)$$

$$i = 1, 2, \ldots, k$$

where $0 = \lambda_o < \lambda_1 < \ldots < \lambda_k < \lambda_{k+1} = 1$ and m satisfies the condition of Lemma 3.1.

Proof: From Lemma 3.1, we know that the point which maximizes K must be in I(0,1). So it satisfies the system of equations

$$\frac{\partial K}{\partial \lambda_i} = \left[\frac{m_i - m_{i-1}}{\lambda_i - \lambda_{i-1}} - \frac{m_{i+1} - m_i}{\lambda_{i+1} - \lambda_i}\right] \cdot M(\lambda_{i-1}, \lambda_i, \lambda_{i+1}) = 0 \ ,$$

$$i = 1, 2, \ldots, k.$$

But from the proof of Lemma 3.1, we know that the first factor in the middle part of the equation is zero when and only when $\lambda_{i-1} = \lambda_i = \lambda_{i+1}$. Such a point cannot provide a maximum for K. This completes the proof.

Theorem 3.1: The system of equations (3.2) has a unique solution $(\lambda_1^C, \ldots, \lambda_k^C)$ in I(0,1) which will give the maximum of K in (3.1) defined on I(0,1) if (3.2) can be rewritten as

$$H_1(y_{i-1}) = H_2(y_i) \ , \quad i = 1, 2, \ldots, k$$

by the substitution

(a) $y_i = h(\lambda_i)/h(\lambda_{i+1})$, $y_o = 0$, $0 \leq y_i < 1$,

$i = 0, 1, \ldots, k,$

or

(b) $y_i = h^*(\lambda_{k-i+1})/h^*(\lambda_{k-i}),\ y_o = 0,\ 0 \leq y_i < 1,$

$i = 0, 1, \ldots, k,$

where $h(\lambda)$ and $h^*(\lambda)$ are both monotonic functions of λ and $H_1(y)$, $H_2(y)$ are as in Lemma 3.2.

Proof: By the assumption of the theorem (3.2) can be rewritten as

$$H_1(y_{i-1}) = H_2(y_i),\quad i = 1, 2, \ldots, k$$

and $y_o = 0$, by Lemma 3.2, y_1 will be uniquely determined, say y_1^C, through the identity $H_2(y_1) = H_1(y_o) = H_1(0)$. We then could obtain a unique y_2, say, y_2^C from $H_2(y_2) = H_1(y_1^C)$. Repeating this process, we have the unique set $(y_1^C, \ldots, y_k^C)$ which satisfies $0 < y_1^C < y_2^C < \ldots < y_k^C < 1$.

We could find the unique spacing $(\lambda_1^C, \lambda_2^C, \ldots, \lambda_k^C)$ from the relation

(a) $h(\lambda_i) = y_i^C \cdot y_{i+1}^C \cdot \ldots \cdot y_k^C \cdot h(\lambda_{k+1}) = y_i^C \cdot y_{i+1}^C \cdot \ldots \cdot y_k^C h(1),$

or

(b) $h^*(\lambda_i) = y_{k-i+1}^C \cdot y_{k-i+2}^C \cdot \ldots \cdot y_k^C h^*(\lambda_o)$

$= y_{k-i+1}^C \cdot y_{k-i+2}^C \cdot \ldots \cdot y_k^C h^*(0).$

The spacing $(\lambda_1^C, \lambda_2^C, \ldots, \lambda_k^C)$ is the optimum spacing for the ABLE θ^* in the complete sample case.

(2) Right Censored Sample

If we are given a right censored sample (2.3), in order to determine the optimum spacing, we have to maximize K defined on $\bar{I}(0,\beta)$. There are two possible cases - (i) $\beta \geq \lambda_k^C$, take $(\lambda_1^C, \lambda_2^C, \ldots, \lambda_k^C)$ as the optimum spacing, (ii) $\beta < \lambda_k^C$, for this case, we have

Lemma 3.4: The function K of (3.1) defined on $\bar{I}(0,\beta)$ with $\beta < \lambda_k^C$ attains its maximum at a point, say, $(\lambda_1^R, \lambda_2^R, \ldots, \lambda_k^R)$ in $\bar{I}(0,\beta)$ with $0<\lambda_1^R<\lambda_2^R<\ldots<\lambda_k^R = \beta$.

Proof: Assume that K attains its maximum at an interior point of $\bar{I}(0,\beta)$. Then this point gives K a relative maximum when it is considered as defined on $\bar{I}(0,1)$. This contradicts the uniqueness of the relative maximum on $\bar{I}(0,1)$. Therefore, the relative maximum must occur on the boundary of $\bar{I}(0,\beta)$. From the proof of Lemma 3.1, we know that $\lambda_{i-1} \neq \lambda_i$ for $i = 1, 2, \ldots, k$. Hence we must have $\lambda_k^R = \beta$.

Theorem 3.2: The spacing $(\lambda_1^R, \lambda_2^R, \ldots, \lambda_k^R)$ is the unique solution of the system of equations

$$M(\lambda_{i-1}, \lambda_i, \lambda_{i+1}) = 0, \quad i = 1,2,\ldots,(k-1), \tag{3.3}$$

with $0 = \lambda_o < \lambda_1 < \lambda_2 < \ldots < \lambda_k = \beta < 1$ if (3.3) can be rewritten as

$$H_1(y_{i-1}) = H_2(y_i), \quad i = 1, 2, \ldots, (k-1),$$

by the substitution

(a) $y_i = h(\lambda_i)/h(\lambda_{i+1})$, $i = 0, 1, \ldots, (k-1)$,

$y_o = 0$, $0 \leq y_i < 1$,

or

(b) $y_i = h^*(\lambda_{k-i+1})/h^*(\lambda_{k-i})$, $i = 1, 2, \ldots, k$,

$0 < y_i < 1$,

where $h(\lambda)$ and $h^*(\lambda)$ are both monotonic functions of λ and $H_1(y)$, $H_2(y)$ are as in Lemma 3.2.

Proof: The proof for case (a) is similar to Theorem 3.1 except that

$$h(\lambda_i) = y_i^C \cdot y_{i+1}^C \cdot \ldots \cdot y_{k-1}^C h(\lambda_k) = y_i^C \cdot y_{i+1}^C \cdot \ldots \cdot y_{k-1}^C h(\beta)$$

is used to obtain the unique spacing $(\lambda_1^R, \lambda_2^R, \ldots, \lambda_k^R)$. And for case (b), most of the arguments are similar to those of the proof of Theorem 3.1 except for the procedure of finding $(y_1^C, \ldots, y_k^C)$, since we do not have $y_o = 0$ here. However, to find the set $(y_1^C, y_2^C, \ldots, y_k^C)$ we can start with any y_1^C and shift y_1^C gradually until the relation $y_1^C\, y_2^C \ldots y_k^C = h^*(\lambda_k)/h^*(\lambda_o) = h^*(\beta)/h^*(0)$ is satisfied. We then could find the unique $(\lambda_1^R, \lambda_2^R, \ldots, \lambda_k^R)$ from the relation $h^*(\lambda_i) = (y_1^C\, y_2^C \ldots y_{k-i}^C)^{-1} h^*(\beta)$.

The spacing $(\lambda_1^R, \lambda_2^R, \ldots, \lambda_k^R)$ is the optimum spacing for the ABLE θ^* in the right censored sample case.

(3) Left Censored Sample

If we are given a left censored sample (2.4), in order to determine the optimum spacing, we have to maximize K defined on $\bar{I}(\alpha, 1)$. Similar to the right censored sample case, there are two possible cases (i) $\alpha \leq \lambda_1^C$, take $(\lambda_1^C, \lambda_2^C, \ldots, \lambda_k^C)$ as the optimum spacing, (ii) $\alpha > \lambda_1^C$, for this case, we have the following lemma whose proof is similar to that of Lemma 3.4:

Lemma 3.5: The function K in (3.1) defined on $\bar{I}(\alpha, 1)$ with $\alpha > \lambda_1^C$ attains its maximum at a point, say, $(\lambda_1^L, \lambda_2^L, \ldots, \lambda_k^L)$ in $\bar{I}(\alpha, 1)$ with $\alpha = \lambda_1^L < \lambda_2^L < \ldots < \lambda_k^L < 1$.

Theorem 3.3: The spacing $(\lambda_1^L, \lambda_2^L, \ldots, \lambda_k^L)$ is the unique solution of the system of equations

$$M(\lambda_{i-1}, \lambda_i, \lambda_{i+1}) = 0, \quad i = 2, 3, \ldots, k, \qquad (3.4)$$

with $0 < \alpha = \lambda_1 < \lambda_2 < \ldots < \lambda_k < \lambda_{k+1} = 1$, if (3.4) can be rewritten as

$$H_1(y_{i-1}) = H_2(y_i), \qquad i = 2, 3, \ldots, k$$

by the substitution

(a) $y_i = h^*(\lambda_{k-i+1})/h^*(\lambda_{k-i})$, $i = 0, 1, \ldots, (k-1)$,

$y_o = 0, \quad 0 \le y_i < 1.$

or

(b) $y_i = h(\lambda_i)/h(\lambda_{i+1})$, $i = 1, 2, \ldots, k$,

$0 < y_i < 1$,

where $h(\lambda)$ and $h^*(\lambda)$ are both monotonic functions of λ and $H_1(y)$, $H_2(y)$ are as in Lemma 3.2.

Proof: The proof for case (a) is similar to Theorem 3.1 except that

$$h^*(\lambda_i)=y^C_{k-i+1}\cdot y^C_{k-i+2}\cdot\ldots\cdot y^C_{k-1}h^*(\lambda_1)=y^C_{k-i+1}\cdot y^C_{k-i+2}\cdot\ldots\cdot y^C_{k-1}h^*(\alpha)$$

is used to obtain the unique spacing $(\lambda^L_1, \lambda^L_2, \ldots, \lambda^L_k)$. And for case (b), most of the arguments are similar to those of the proof of Theorem 3.1 except that in the procedure for finding $(y^C_1, y^C_2, \ldots, y^C_k)$, we do not have $y_o = 0$ here. However, to find the set $(y^C_1, y^C_2, \ldots, y^C_k)$ we can start with any y^C_1 and shift y^C_1 gradually until the relation $y^C_1\cdot y^C_2\cdot\ldots\cdot y^C_k = h(\lambda_1)/h(\lambda_{k+1}) = h(\alpha)/h(1)$ is satisfied. We then could find the unique spacing $(\lambda^L_1, \lambda^L_2, \ldots, \lambda^L_k)$ from the relation $h(\lambda_i) = (y^C_1\cdot y^C_2\cdot\ldots\cdot y^C_{i-1})^{-1}h(\alpha)$.

The spacing $(\lambda^L_1, \lambda^L_2, \ldots, \lambda^L_k)$ is the optimum spacing for the ABLE θ^* in the left censored sample case.

(4) Doubly Censored Sample

If we are given a doubly censored sample (2.5), in order to determine the optimum spacing, we have to maximize K defined on $\overline{I}(\alpha, \beta)$. There are five possible cases

(i) $\alpha \leq \lambda_1^C$ and $\beta \geq \lambda_k^C$, take $(\lambda_1^C, \ldots, \lambda_k^C)$ as the optimum spacing,

(ii) $\alpha \leq \lambda_1^R$ and $\beta = \lambda_k^R < \lambda_k^C$, take $(\lambda_1^R, \ldots, \lambda_k^R)$ as the optimum spacing,

(iii) $\lambda_1^C < \alpha = \lambda_1^L$ and $\beta \geq \lambda_k^L$, take $(\lambda_1^L, \ldots, \lambda_k^L)$ as the optimum spacing,

(iv) $\alpha > \lambda_1^R$ and $\beta = \lambda_k^R < \lambda_k^C$,

(v) $\lambda_1^C < \alpha = \lambda_1^L$ and $\beta < \lambda_k^L$.

For cases (iv) and (v) we have

Lemma 3.6: The function K in (3.1) defined on $\overline{I}(\alpha, \beta)$ with either (i) $\alpha > \lambda_1^R$ and $\beta = \lambda_k^R < \lambda_k^C$ or (ii) $\lambda_1^C < \alpha = \lambda_1^L$ and $\beta > \lambda_k^L$ attains its maximum at a point, say, $(\lambda_1^D, \lambda_2^D, \ldots, \lambda_k^D)$ in $\overline{I}(\alpha, \beta)$ with $\alpha = \lambda_1^D < \lambda_2^D < \ldots < \lambda_k^D = \beta$.

Proof: We will give the proof for case (i) here. Case (ii) can be proved in a similar way.

First we show that if $\alpha > \lambda_1^R$ and $\beta = \lambda_k^R < \lambda_k^C$, then K cannot have a maximum at an interior point of $\overline{I}(\alpha, \beta)$.

Assume that K attains its maximum at an interior point $(\lambda_1^*, \ldots, \lambda_k^*)$ of $\overline{I}(\alpha, \beta)$. Then this point gives K on $\overline{I}(\alpha, \beta)$ a relative maximum and hence must satisfy (3.2) with $\alpha < \lambda_1^*$ and $\beta > \lambda_k^*$.

This solution is different from $(\lambda_1^C, \ldots, \lambda_k^C)$ since $\lambda_k^* < \beta < \lambda_k^C$. But this contradicts that the solution for (3.2) is unique. So the maximum of K on $\overline{I}(\alpha, \beta)$ occurs at a boundary point $(\lambda_1^D, \ldots, \lambda_k^D)$, i.e. $\lambda_i^D = \lambda_{i+1}^D$ for some i, $i = 1, 2, \ldots, k-1$. Following arguments similar to those in the proof of Lemma 3.1, it can be shown that $\lambda_i^D < \lambda_{i+1}^D$ for $i = 1, 2, \ldots, k-1$. So there are three possibilities:

(1) $\alpha = \lambda_1^D$ and $\lambda_k^D = \beta$;

(2) $\alpha < \lambda_1^D$ and $\lambda_k^D = \beta$;

(3) $\alpha = \lambda_1^D$ and $\lambda_k^D < \beta$.

For possibility (1), we can write $K = K(\alpha, \lambda_2, \lambda_3, \ldots, \lambda_{k-1}, \beta) = Q_1 + m^2(\alpha)/\alpha + m^2(\beta)/(1-\beta)$, where $Q_1 = Q_1(\lambda_2, \ldots, \lambda_{k-1}) = \sum_{i=2}^{k} (m_i - m_{i-1})^2/(\lambda_i - \lambda_{i-1})$ with $\lambda_1 = \alpha$ and $\lambda_k = \beta$. Since $(\alpha, \lambda_2^D, \ldots, \lambda_{k-1}^D, \beta)$ maximizes K, $(\lambda_2^D, \ldots, \lambda_{k-1}^D)$ maximizes Q_1. Following arguments similar to those in the proof of Lemma 3.1 (replacing 0 by α, 1 by β and k by k-2), when defined on the domain $\alpha \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_{k-1} \leq \beta$, Q_1 attains its maximum at an interior point $(\lambda_2^D, \ldots, \lambda_{k-1}^D)$ of the domain.

For possibility (2), first let us write $K = K(\lambda_1, \ldots, \lambda_{k-1}, \beta) = Q_2 + m^2(\beta)/(1-\beta)$, where $Q_2 = \sum_{i=1}^{k} (m_i - m_{i-1})^2/(\lambda_i - \lambda_{i-1})$ with $\lambda_k = \beta$. Any point of the form $(\lambda_1, \ldots, \lambda_{k-1}, \beta)$ which maximizes this K must also maximize Q_2. We already know that $\alpha < \lambda_1^D < \lambda_2^D < \ldots < \lambda_{k-1}^D < \beta$. So this point is a solution for the system of equations like those of Theorem 3.2. This solution is different from $(\lambda_1^R, \ldots, \lambda_k^R)$, since $\lambda_1^R < \alpha < \lambda_1^D$. This contradicts that $(\lambda_1^R, \ldots, \lambda_k^R)$ is the unique solution. So possibility is not valid.

Similar conclusions can be drawn for possibility (3).

Theorem 3.4: The spacing $(\lambda_1^D, \ldots, \lambda_k^D)$ is the unique solution of the system of equations

$$M(\lambda_{i-1}, \lambda_i, \lambda_{i+1}) = 0, \qquad i = 2, 3, \ldots, (k-1), \quad (3.5)$$

with $0 < \alpha = \lambda_1 < \lambda_2 < \ldots < \lambda_k = \beta < 1$, if (3.5) can be rewritten as

$$H_1(y_{i-1}) = H_2(y_i), \qquad i = 2, 3, \ldots, (k-1),$$

by the substitution

(a) $y_i = h(\lambda_i)/h(\lambda_{i+1})$, $0 < y_i < 1$, $i=1,2,\ldots,(k-1)$,

or

(b) $y_i = h^*(\lambda_{k-i+1})/h^*(\lambda_{k-i})$, $0<y_i<1$, $i=1,2,\ldots,(k-1)$,

where $h(\lambda)$ and $h^*(\lambda)$ are both monotonic functions of λ and $H_1(y)$, $H_2(y)$ are as in Lemma 3.2.

Proof: Since (3.5) can be rewritten as

$$H_1(y_{i-1}) = H_2(y_i), \quad i = 2, 3, \ldots, k-1$$

to find the set $(y_1^C, \ldots, y_{k-1}^C)$ we start with any y_1^C and shift y_1^C gradually until the relations

(a) $y_1^C \cdot y_2^C \cdot \ldots \cdot y_{k-1}^C = h(\lambda_1)/h(\lambda_k) = h(\alpha)/h(\beta)$

or

(b) $y_1^C \cdot y_2^C \cdot \ldots \cdot y_{k-1}^C = h^*(\lambda_k)/h^*(\lambda_1) = h^*(\beta)/h^*(\alpha)$

is satisfied. We then could find the unique spacing $(\lambda_1^D, \lambda_2^D, \ldots, \lambda_k^D)$ from the relation

(a) $h(\lambda_i) = (y_1^C \cdot y_2^C \cdot \ldots \cdot y_{i-1}^C)^{-1} h(\alpha)$

(or) $= y_i^C \cdot y_{i+1}^C \cdot \ldots \cdot y_{k-1}^C h(\beta)$

(b) $h^*(\lambda_i) = (y_1^C \cdot y_2^C \cdot \ldots \cdot y_{k-i}^C)^{-1} h^*(\beta)$

(or) $= y_{k-i+1}^C \cdot y_{k-i+2}^C \cdot \ldots \cdot y_{k-i}^C h^*(\alpha)$.

The spacing $(\lambda_1^D, \lambda_2^D, \ldots, \lambda_k^D)$ is the optimum spacing for the ABLE θ^* in a doubly censored sample case.

CHAPTER 4

THE ABLE OF THE LOCATION PARAMETER OF THE LOGISTIC DISTRIBUTION

4.1 Introduction

Let a given ordered sample be from a logistic distribution with probability density function (pdf)

$$\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} \frac{\exp[-(x-\mu)/\sigma]}{\{1+\exp[-(x-\mu)/\sigma]\}^2} ,$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$. The value of the scale parameter σ is assumed to be known and the value of the location parameter μ is to be estimated. The cumulative distribution function (cdf) is

$$F\left(\frac{x-\mu}{\sigma}\right) = \{1+\exp[-\frac{x-\mu}{\sigma}]\}^{-1} .$$

Let $U = (X-\mu)/\sigma$ be the standarized random variable. Then U has the pdf and cdf, respectively,

$$f(u) = \frac{\exp(-u)}{[1+\exp(-u)]^2} ,$$

$$F(u) = [1+\exp(-u)]^{-1} .$$

The usefulness of the logistic distribution is in studies of population growth, of life test data, of bio-assay and of biochemical data, etc.

Many authors have considered estimation of the parameters of the logistic distribution by various methods. Among them are Berkson (1957), Gupta, Qureishi and Shah (1965), Harter and Moore (1967), Tiku (1968), Hassanein (1969), Chan (1969), Gupta and Gnanadesikan (1966), Chan, Chan and Mead (1971), Chan and Cheng (1972).

Both Gupta and Gnanadesikan (1966), and Chan (1969) have provided the optimum spacing for the ABLE μ^* when σ is known only for the complete sample case. We will use our unified method as stated in Chapter 3 which is different from theirs to derive the unique optimum spacing for the ABLE μ^* when σ is known based on complete, right, left and doubly censored samples. Later on, in a separate chapter we will give the algorithm to obtain an optimum spacing for the ABLE μ^* when σ is known based on a multiply censored sample.

In the case of the logistic distribution, the following notation (corresponding to that in previous chapters) will be useful in obtaining the ABLE μ^* when σ is known:

$$f_i = f(u_i) = \lambda_i(1-\lambda_i), \quad \sum = \sum_{i=1}^{k+1},$$

$$f_i u_i = \lambda_i(1-\lambda_i)[\ln \lambda_i - \ln(1-\lambda_i)],$$

$$K_1 = \sum(\lambda_i-\lambda_{i-1})(1-\lambda_i-\lambda_{i-1})^2,$$

$K_3 = \sum(1-\lambda_i-\lambda_{i-1})(f_i u_i - f_{i-1}u_{i-1})$,

$a_i = \lambda_i(1-\lambda_i)(\lambda_{i+1}-\lambda_{i-1})/K_1$,

$\lambda_o = 0,\ \lambda_{k+1} = 1,\quad f_o = f_{k+1} = f_o u_o = f_{k+1}u_{k+1} = 0,$

$$\mu^* = \sum_{i=1}^{k} a_i x_{(n_i)} - \sigma \frac{K_3}{K_1} ,$$

$$A\ V\ (\mu^*) = \frac{\sigma^2}{n} \cdot \frac{1}{K_1} ,$$

$ARE(\mu^*) = 3K_1$ = asymptotic efficiency relative to the Cramér-Rao lower bound.

In order to determine the optimum spacing, we need the following notation to prove the lemmas and theorems:

$$\theta = \mu,\ \theta^* = \mu^*,\quad K = K_1 \quad m_i = f_i \quad .$$

The Lemma 3.1 is established by the fact that $d^2m/d\lambda^2 = -2 \neq 0$.

4.2 Complete Sample (c '. (2.1))

Theorem 4.1: There is one and only one optimum spacing $\{\lambda_i^C\}_L = (\lambda_1^C, \lambda_2^C, \ldots, \lambda_k^C)$ for the ABLE μ^* when σ is known, based on k order statistics selected from a complete sample.

Proof: Lemma 3.2 and Theorem 3.1 follow from the substitutions

$$h(\lambda) = \lambda, \text{ (i.e. } y_i = \lambda_i/\lambda_{i+1},\ i = 0,1,\ldots,k),$$

$$H_1(y) = 1-y, \quad H_2(y) = (1-y)/y \ .$$

The completion of the proof follows from Lemmas 3.1, 3.2, 3.3 and Theorem 3.1.

Algorithm:

Solving the following system of equations

$$H_1(y_{i-1}) = H_2(y_i), \quad i = 1, 2, \ldots, k$$

with $y_o = 0$, we have

$$(y_1^C, \ldots, y_k^C) = (1/2, 2/3, \ldots, i/(i+1),\ldots,k/(k+1)),$$

$$\lambda_i^C = h(\lambda_i^C) = y_i^C \cdot y_{i+1}^C \cdot \ldots \cdot y_k^C h(1) = i/(k+1),$$

$i = 1, 2, \ldots, k$

Hence $\{\lambda_i^C\}_L = \{i/(k+1)\}$.

4.3 Singly Censored Sample:

(a) Right censored sample (cf (2.3))

Theorem 4.2: There is one and only one optimum spacing $\{\lambda_i^R\}_L = (\lambda_1^R, \lambda_2^R, \ldots, \lambda_k^R)$ for the ABLE μ^* when σ is known, based on k order statistics selected from a right censored sample.

Proof: (i) If $\beta \geq \lambda_k^C$, then $\{\lambda_i^R\}_L = \{\lambda_i^C\}_L$, (ii) if $\beta < \lambda_k^C$, the theorem follows from Lemmas 3.1, 3.2, 3.4 and Theorem 3.2 by the substitutions

$$h(\lambda) = \lambda, \quad H_1(y) = 1-y, \; H_2(y) = (1-y)/y.$$

Algorithm:

(i) If $\beta \geq \lambda_k^C = k/(k+1)$, take $\{\lambda_i^R\}_L = \{\lambda_i^C\}_L$.

(ii) If $\beta < \lambda_k^C = k/(k+1)$, solve the following system of equations

$$H_1(y_{i-1}) = H_2(y_i), \; i = 1, 2, \ldots, k-1.$$

with $y_o = 0$, to obtain

$$(y_1^C, \ldots, y_{k-1}^C) = (1/2, 2/3, \ldots, i/(i+1), \ldots, (k-1)/k),$$

$$\lambda_i^R = h(\lambda_i^R) = y_i^C \cdot y_{i+1}^C \cdot \ldots \cdot y_{k-1}^C h(\beta) = i\beta/k, \; i=1,2,\ldots,(k-1),$$

$$\lambda_k^R = \beta \; .$$

Hence $\{\lambda_i^R\}_L = \{i\beta/k\}$.

(B) Left censored sample (cf. (2.4))

Theorem 4.3: There is one and only one optimum spacing $\{\lambda_i^L\}_L = (\lambda_1^L, \lambda_2^L, \ldots, \lambda_k^L)$ for the ABLE μ^* when σ is known, based on k order statistics selected from a left censored sample.

Proof: (i) If $\alpha \leq \lambda_1^C$, then $\{\lambda_i^L\}_L = \{\lambda_i^C\}_L$, (ii) if $\alpha > \lambda_1^C$, the theorem follows from Lemmas 3.1, 3.2, 3.5 and Theorem 3.3 by the substitutions

$$h(\lambda) = \lambda, \; H_1(y) = 1-y, \; H_2(y) = (1-y)/y \; .$$

Algorithm:

(i) If $\alpha \leq \lambda_1^C = 1/(k+1)$, take $\{\lambda_i^L\}_L = \{\lambda_i^C\}_L$.

(ii) If $\alpha > \lambda_1^C = 1/(k+1)$, solve the following system of equations

$$H_1(y_{i-1}) = H_2(y_i), \quad i = 2, 3, \ldots, k.$$

If we start from $y_1^C = a$ (a fixed constant between 0 and 1), then we have

$$y_i^C = \frac{(i-1)-(i-2)a}{i-(i-1)a}, \quad i = 1, 2, \ldots, k \; .$$

These have to satisfy the relation

$$y_1^C \cdot y_2^C \cdot \ldots \cdot y_k^C = \lambda_1^L/\lambda_{k+1}^L = \alpha \; .$$

This implies $a = \alpha k/[\alpha(k-1)+1]$,

$$y_i^C = \frac{(i-1)(1-\alpha)+\alpha k}{i(1-\alpha)+\alpha k}, \quad i = 1, 2, \ldots, k.$$

Hence

$$\lambda_i^L = h(\lambda_i^L) = (y_1^C \cdot y_2^C \cdot \ldots \cdot y_{i-1}^C)^{-1} h(\alpha) = \frac{(i-1)(1-\alpha)}{k} + \alpha,$$

$i = 2, 3, \ldots, k,$ and $\lambda_1^L = \alpha$.

Therefore $\{\lambda_i^L\}_L = \{\frac{i-1}{k}(1-\alpha) + \alpha\}$.

4.4 Doubly Censored Sample (cf. (2.5))

Theorem 4.4: There is one and only one optimum spacing $\{\lambda_i^D\}_L = (\lambda_1^D, \lambda_2^D, \ldots, \lambda_k^D)$ for the ABLE μ^* when σ is known, based on k order statistics selected from a doubly censored sample.

Proof: (i) If $\alpha \leq \lambda_1^C = 1/(k+1)$ and $\beta \geq \lambda_k^C = k/(k+1)$, then $\{\lambda_i^D\}_L = \{\lambda_i^C\}_L$, (ii) if $\alpha \leq \lambda_1^R = \beta/k$ and $\beta < \lambda_k^C = k/(k+1)$, then $\{\lambda_i^D\}_L = \{\lambda_i^R\}_L$, (iii) if $\alpha > \lambda_1^C = 1/(k+1)$ and $\beta \geq \lambda_k^L = (k-1+\alpha)/k$, then $\{\lambda_i^D\}_L = \{\lambda_i^L\}_L$.

(iv) If $\alpha > \lambda_1^R = \beta/k$ and $\beta < \lambda_k^C = k/(k+1)$ or (v) if $\alpha > \lambda_1^C = 1/(k+1)$ and $\beta < \lambda_k^L = (k-1+\alpha)/k$, then the theorem follows from Lemmas 3.1, 3.2, 3.6 and Theorem 3.4 by the substitutions

$$h(\lambda) = \lambda, \quad H_1(y) = 1-y, \; H_2(y) = (1-y)/y \; .$$

Algorithm:

(i) If $\alpha \leq \lambda_1^C = 1/(k+1)$ and $\beta > \lambda_k^C = k/(k+1)$, take $\{\lambda_i^D\}_L = \{\lambda_i^C\}_L$.

(ii) If $\alpha \leq \lambda_1^R = \beta/k$ and $\beta < \lambda_k^C = k/(k+1)$, take

$$\{\lambda_i^D\}_L = \{\lambda_i^R\}_L \; .$$

(iii) If $\alpha > \lambda_1^C = 1/(k+1)$ and $\beta \geq \lambda_k^L = (k-1+\alpha)/k$,

take $\{\lambda_i^D\}_L = \{\lambda_i^L\}_L$.

(iv) If $\alpha > \lambda_1^R = \beta/k$ and $\beta < \lambda_k^C = k/(k+1)$, solve the following system of equations

$$H_1(y_{i-1}) = H_2(y_i),\ i = 2, 3, \ldots, (k-1).$$

If we start from $y_1^C = a$ (a fixed constant between 0 and 1), then we have

$$y_i^C = \frac{(i-1)-(i-2)a}{i-(i-1)a}\ ,\ i = 1, 2, \ldots, (k-1)\ .$$

These have to satisfy the relation

$$y_1^C \cdot y_2^C \cdot \ldots \cdot y_{k-1}^C = \lambda_1^D / \lambda_k^D = \alpha/\beta\ ,$$

which implies that, $a = (k-1)\alpha/[\beta+(k-2)\alpha]$,

$$y_i^C = \frac{(i-1)\beta+(k-i)\alpha}{i\beta+(k-i-1)\alpha}\ , \quad i = 1, 2, \ldots, (k-1).$$

Hence $\lambda_i^D = h(\lambda_i^D) = (y_1^C \cdot y_2^C \cdot \ldots \cdot y_{i-1}^C)^{-1} h(\alpha)$

$$= \frac{(k-i)\alpha+(i-1)\beta}{k-1}\ , \quad i = 2, 3, \ldots, k$$

and $\lambda_1^D = \alpha$.

Therefore $\{\lambda_i^D\}_L = \{\frac{(k-i)\alpha+(i-1)\beta}{k-1}\}$.

(v) If $\alpha > \lambda_1^C = 1/(k+1)$ and $\beta < \lambda_k^L = (k-1+\alpha)/k$, then in a way similar to that in (iv), we have

$$\{\lambda_i^D\}_L = \{\frac{(k-i)\alpha+(i-1)\beta}{k-1}\}\ .$$

By substitution of the optimum spacing, we could express the coefficients a_i in the ABLE μ^* and ARE (μ^*) in terms of the optimum spacings. Table II gives the a_i's and ARE(μ^*)'s for all the different cases. Table I lists the expressions for the optimum spacings which were derived above.

4.5 Remarks

We computed the ARE(μ^*) in a few examples for different cases and found out that for small α and large β, the ARE(μ^*) based on left censored (L), right censored (R) and doubly censored (D) samples are quite close to that based on a complete sample (C)

<1> $(k,\alpha,\beta) = (3, .30, .70)$,
ARE(μ^*) of (C, R, L, D) = (.9375, .9349, .9349,.9300),

<2> $(k,\alpha,\beta) = (4,.45,.70)$,
ARE(μ^*) of (C,R,L,D) = (.9600,.9516,.8985,.8801),

<3> $(k,\alpha,\beta) = (.5,.45,.65)$,
ARE(μ^*) of (C,R,L,D) = (.9722,.9461,.9022,.8655).

For the complete samples case, if we compare our results to those obtained by Gupta and Gananadesikan (1966), and Chan (1969), it can be seen that our results have an advantage in finding the optimum spacings, the corresponding coefficients of the ABLE (μ^*), and the ARE (μ^*) more accurately since our results are expressed as formulas.

CHAPTER 5

THE ABLE OF THE SCALE PARAMETER OF THE PARETO DISTRIBUTION

5.1 Introduction

Let a given ordered sample be from a Pareto distribution with pdf

$$\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\gamma\left(1+ \frac{x-\mu}{\sigma}\right)^{-(\gamma+1)} ,$$

where $\mu \leq x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$ and $\gamma > 0$. The values of the location parameter μ and the shape parameter γ are assumed known and the value of the scale parameter σ is to be estimated. The cdf is

$$F\left(\frac{x-\mu}{\sigma}\right) = 1 - \left(1+ \frac{x-\mu}{\sigma}\right)^{-\gamma} .$$

The standardized variable $U = (X-\mu)/\sigma$ has the pdf and cdf, respectively,

$$f(u) = \gamma(1+u)^{-(\gamma+1)} ,$$

$$F(u) = 1 - (1+u)^{-\gamma} .$$

The Pareto distribution was named after a Swiss professor of economics, V. Pareto and can be used to describe distribution of incomes, assets of firms, property values, sizes of the cities and numbers of firms in various industries and so on.

Quandt (1966), Likeš (1969), Aigner and Goldberger (1970), Malik (1970), Y.H. Wang (1971) considered estimation of the scale and/or shape parameters when the location and scale parameters are equal. Moore and Harter (1967, 1969) considered estimation of the shape parameter when the location and scale parameters are known.

Maximum likelihood estimates based on all observations and linear estimates based on selected order statistics of the scale and/or location parameters were considered by Kulldorff and Vännman (1973).

The basic properties and a survey of the literature up to 1970 can be found in Johnson and Kotz (1970).

Kulldorff and Vännman (1973) used a dynamic programming approach to find the optimum spacing for the ABLE σ^* when μ is known for complete samples only. In this chapter, we will use our unified method to obtain the unique optimum spacing for the ABLE σ^* when μ is known based on complete, singly censored and doubly censored samples. Our optimum spacings and the other corresponding results for complete samples coincide with those of Kulldorff and Vännman (1973).

In the case of the Pareto distribution, the following formulas will be used to obtain the ABLE σ^* when μ is known.

$$f_i = f(u_i) = \gamma(1-\lambda_i)^{1+\frac{1}{\gamma}}, \quad \sum = \sum_{i=1}^{k}$$

$$f_i u_i = \gamma(1-\lambda_i) - \gamma(1-\lambda_i)^{\frac{1}{\gamma}+1},$$

$$b_i = \frac{\gamma^2}{K_2}(1-\lambda_i)^{\frac{1}{\gamma}}\left[\frac{(1-\lambda_{i+1})^{\frac{1}{\gamma}}-(1-\lambda_i)^{\frac{1}{\gamma}}}{(1-\lambda_{i+1})^{-1}-(1-\lambda_i)^{-1}} - \frac{(1-\lambda_i)^{\frac{1}{\gamma}}-(1-\lambda_{i-1})^{\frac{1}{\gamma}}}{(1-\lambda_i)^{-1}-(1-\lambda_{i-1})^{-1}}\right],$$

$$K_2 = \gamma^2 \sum \frac{[(1-\lambda_i)^{\frac{1}{\gamma}}-(1-\lambda_{i-1})^{\frac{1}{\gamma}}]^2}{(1-\lambda_i)^{-1}-(1-\lambda_{i-1})^{-1}},$$

$$K_3/K_2 = \sum b_i,$$

$$\lambda_o = 0, \ \lambda_{k+1} = 1, \quad f_o = f_{k+1} = f_o u_o = f_{k+1}u_{k+1} = 0,$$

$$\sigma^* = \sum b_i x_{(n_i)} - \mu\frac{K_3}{K_2},$$

$$A\ V\ (\sigma^*) = \frac{\sigma^2}{n}\cdot\frac{1}{K_2},$$

$ARE(\sigma^*) = (\gamma+2)K_2/\gamma$ = asymptotic efficiency relative to the Cramér-Rao lower bound.

In order to determine the optimum spacing, we use the following notation to prove the lemmas and theorems:

$$\theta = \sigma, \quad \theta^* = \sigma^*, \quad K = K_2, \quad m_i = f_i u_i \quad .$$

The Lemma 3.1 is established by the fact that $d^2m/d\lambda^2 = -(\gamma+1)(1-\lambda)^{\frac{1}{\gamma}-1}/\gamma \neq 0$.

5.2 Complete Sample (cf. (2.1))

Theorem 5.1: There is one and only one optimum spacing $\{\lambda_i^C\}_P = (\lambda_1^C, \lambda_2^C, \ldots, \lambda_k^C)$ for the ABLE σ^* when μ is known, based on k order statistics selected from a complete sample.

Proof: Lemma 3.2 and Theorem 3.1 follow from the substitutions:

$h^*(\lambda) = (1-\lambda)^{\frac{1}{\gamma}}$, (i.e. $y_i = (1-\lambda_{k-i+1})^{\frac{1}{\gamma}}/(1-\lambda_{k-i})^{\frac{1}{\gamma}}$,

$i = 0, 1, \ldots, k$), $H_1(y) = \gamma(y^{\gamma+1}-y^{\gamma})/[2(1-y^{\gamma})]+1/2$,

$H_2(y) = -1/2 - \gamma(y-1)/[2(y-y^{\gamma+1})]$.

Then the theorem follows from Lemmas 3.1, 3.2, 3.3 and Theorem 3.1.

Algorithm:

Solving the following system of equations

$$H_1(y_{i-1}) = H_2(y_i), \quad i = 1, 2, \ldots, k$$

with $y_o^C = 0$, we obtain the unique set $(y_1^C, y_2^C, \ldots, y_k^C)$. Then we can find the unique spacing $\{\lambda_i^C\}_P$ from the relation

$$\lambda_i^C = 1-(y_{k-i+1}^C \cdot y_{k-i+2}^C \cdot \ldots \cdot y_k^C)^\gamma, \quad i=1,2,\ldots,k.$$

Table III lists all the values of y_i^C for $\gamma = .5(.5)3.0(1.0)10.0$ and $k = 1(1)20$.

5.3 Singly Censored Sample

(A) Right censored sample (cf. (2.3))

Theorem 5.2: There is one and only one optimum spacing $\{\lambda_i^R\}_P = (\lambda_1^R, \lambda_2^R, \ldots, \lambda_k^R)$ for the ABLE σ^* when μ is known, based on k order statistics selected from a right censored sample.

Proof: (i) If $\beta \geq \lambda_k^C$, then $\{\lambda_i^R\}_P = \{\lambda_i^C\}_P$,

(ii) if $\beta < \lambda_k^C$, the theorem follows from Lemmas 3.1, 3.2, 3.4 and Theorem 3.2 by the substitutions

$$h^*(\lambda) = (1-\lambda)^{\frac{1}{\gamma}}, \quad H_1(y) = \gamma(y^{\gamma+1}-y^\gamma)/[2(1-y^\gamma)]+ 1/2,$$

$$H_2(y) = -1/2 - \gamma(y-1)/[2(y-y^{\gamma+1})].$$

Algorithm:

(i) If $\beta \geq \lambda_k^C$, take $\{\lambda_i^R\}_P = \{\lambda_i^C\}_P$.

(ii) If $\beta < \lambda_k^C$, use numerical iteration to obtain $(y_1^C, y_2^C, \ldots, y_k^C)$ starting from $y_1^C = .5$ and shifting y_1^C to the right or the left of .5 successively until the relation

$$y_1^C \cdot y_2^C \cdot \ldots \cdot y_k^C = (1-\beta)^{\frac{1}{\gamma}}$$

is satisfied. Then

$$\lambda_i^R = \beta,\ \lambda_i^R = 1-(1-\beta)(y_1^C \cdot y_2^C \cdot \ldots \cdot y_{k-i}^C)^{-\gamma},$$

$i = 1, 2, \ldots, (k-1)$.

(B) Left censored sample (cf. (2.4))

Theorem 5.3: There is one and only one optimum spacing $\{\lambda_i^L\}_P = (\lambda_1^L, \lambda_2^L, \ldots, \lambda_k^L)$ for the ABLE σ^* when μ is known, based on k order statistics selected from a left censored sample.

Proof: (i) If $\alpha \leq \lambda_1^C$, then $\{\lambda_i^L\}_P = \{\lambda_i^C\}_P$, (ii) if $\alpha > \lambda_1^C$, the theorem follows from Lemmas 3.1, 3.2, 3.5 and Theorem 3.3 by the substitutions

$$h^*(\lambda) = (1-\lambda)^{\frac{1}{\gamma}},\ H_1(y) = \gamma(y^{\gamma+1}-y^\gamma)/[2(1-y^\gamma)] + 1/2,$$

$$H_2(y) = -1/2 - \gamma(y-1)/[2(y-y^{\gamma+1})].$$

Algorithm:

(i) If $\alpha \leq \lambda_1^C$, take $\{\lambda_i^L\}_P = \{\lambda_i^C\}_P$.

(ii) If $\alpha > \lambda_1^C$, solve the following system of equations

$$H_1(y_{i-1}) = H_2(y_i),\ i = 1, 2, \ldots, (k-1)$$

with $y_o^C = 0$ to obtain the unique set $(y_1^C, y_2^C, \ldots, y_{k-1}^C)$ which is coincident with the complete sample case. We therefore have the unique spacing $\{\lambda_i^L\}$ given by

$$\lambda_1^L = \alpha, \ \lambda_i^L = 1-(1-\alpha)(y_{k-i+1}^C \cdot y_{k-i+2}^C \cdot \ldots \cdot y_{k-1}^C)^\gamma,$$

$$i = 2, 3, \ldots, k.$$

The values of y_i^C in Table III can be applied here.

5.4 Doubly Censored Sample (cf. (2.5))

Theorem 5.4: There is one and only one optimum spacing $\{\lambda_i^D\}_P = (\lambda_1^D, \lambda_2^D, \ldots, \lambda_k^D)$ for the ABLE σ^* when μ is known, based on k order statistics selected from a doubly censored sample.

Proof: (i) If $\alpha \leq \lambda_1^C$ and $\beta > \lambda_k^C$, then $\{\lambda_i^D\}_P = \{\lambda_i^C\}_P$, (ii) if $\alpha \leq \lambda_1^R$ and $\beta < \lambda_k^C$, then $\{\lambda_i^D\}_P = \{\lambda_i^R\}_P$, (iii) if $\alpha > \lambda_1^C$ and $\beta \geq \lambda_k^L$, then $\{\lambda_i^D\}_P = \{\lambda_i^L\}_P$. (iv) If $\alpha > \lambda_1^R$ and $\beta < \lambda_k^C$ or (v) if $\alpha > \lambda_1^C$ and $\beta < \lambda_k^L$, the theorem follows from Lemmas 3.1, 3.2, 3.6 and Theorem 3.4 by the substitutions

$$h^*(\lambda) = (1-\lambda)^{\frac{1}{\gamma}}, \ H_1(y) = \gamma(y^{\gamma+1}-y^\gamma)/[2(1-y^\gamma)]+ 1/2,$$

$$H_2(y) = - 1/2 - \gamma(y-1)/[2(y-y^{\gamma+1})].$$

Algorithm:

(i) If $\alpha \leq \lambda_1^C$ and $\beta > \lambda_k^C$, take $\{\lambda_i^D\}_P = \{\lambda_i^C\}_P$.

(ii) If $\alpha \leq \lambda_1^R$ and $\beta < \lambda_k^C$, take $\{\lambda_i^D\}_P = \{\lambda_i^R\}_P$.

(iii) If $\alpha > \lambda_1^C$ and $\beta \geq \lambda_k^C$, take $\{\lambda_i^D\}_P = \{\lambda_i^L\}_P$.

(iv) If $\alpha > \lambda_1^R$ and $\beta < \lambda_k^C$ or

(v) if $\alpha > \lambda_1^C$ and $\beta < \lambda_k^L$, use numerical iteration to obtain $(y_1^C, y_2^C, \ldots, y_{k-1}^C)$, starting from $y_1^C = .5$ and shifting y_1^C to the right or to the left of .5 successively until the relation

$$y_1^C \cdot y_2^C \cdot \ldots \cdot y_{k-1}^C = [(1-\beta)/(1-\alpha)]^{\frac{1}{\gamma}}$$

is satisfied. Then the unique spacing $\{\lambda_i^D\}_P$ is obtained as follows:

$$\lambda_1^D = \alpha, \; \lambda_k^D = \beta, \; \lambda_i^D = 1-(1-\alpha)(y_{k-i+1}^C \cdot y_{k-i+2}^C \cdot \ldots \cdot y_{k-1}^C)^{\gamma}$$

or $\lambda_i^D = 1-(1-\beta)(y_1^C \cdot y_2^C \cdot \ldots \cdot y_{k-i}^C)^{-\gamma}$, $i = 2, 3, \ldots, (k-1)$.

Tables IV and V provide values of some of the optimum spacings $\{\lambda_i^R\}_P$, $\{\lambda_i^D\}_P$, the corresponding coefficients b_i and the ARE (σ^*) of the ABLE σ^* when $\gamma = 2.0(1.0)5.0$ based on $k = 2(1)5$ order statistics from right censored and doubly censored samples respectively, for various censoring proportions.

5.5 Remarks

The following example illustrates how to use Table III.

Example: Suppose $\gamma = 1.0$, $k = 4$

(a) Complete sample

From Table III

$$y_1^C = .500000,\ y_2^C = .666667,\ y_3^C = .750000,\ y_4^C = .800000.$$

$$\text{Hence } \{\lambda_i^C\}_P = \{1-(y_4^C\cdot\ldots\cdot y_{4-i+1}^C)^{1.0},\quad i = 1,\ \ldots,\ 4\}$$

$$= \{.200000,\ .400000,\ .600000,\ .800000\}$$

Using the formula to find the b_i's,

$$\{b_i\} = \{.800000,\ .450000,\ .200000,\ .050000\}$$

$$K_3/K_2 = 1.500000$$

$$\text{ARE }(\sigma^*) = .960000$$

(b) Left censored sample with $\alpha = .30$

The values of y_i's are the same as those in (a).

$$\lambda_1^L = \alpha = .30,\ \{\lambda_i^L,\ i=2,3,4\}_P = \{1-(1-.30)(y_{4-1}\ \cdots\ y_{4-i+1})^{1.0},$$

$$i = 2,3,4\} = \{.475000,\ .650000,\ .825000\}$$

$\{b_i\} = \{.733793, .304138, .135172, .033793\}$,

$K_3/K_2 = 1.206897$,

ARE $(\sigma^*) = .951563$.

FORTRAN IV programs were written for finding the values of the optimum spacings, the corresponding coefficients and the ARE for both right censoring and double censoring under any censoring proportions and any values of k and γ. Programs for finding the values of y_i^c for any value of k and γ have also been written.

CHAPTER 6

THE ABLE OF THE SCALE

PARAMETER OF THE RAYLEIGH DISTRIBUTION

6.1 Introduction

Let a given ordered sample be from a Rayleigh distribution with pdf

$$\frac{1}{\sigma} f(\frac{x}{\sigma}) = \frac{x}{\sigma^2} \exp(- \frac{x^2}{2\sigma^2}), \sigma > 0, x > 0,$$

where σ is called the scale parameter of the distribution and is to be estimated. The cdf is

$$F(\frac{x}{\sigma}) = 1 - \exp(- \frac{x^2}{2\sigma^2}) .$$

The standardized variable $U = x/\sigma$ has the pdf and cdf, respectively,

$$f(u) = ue^{-u^2/2}$$

$$F(u) = 1-e^{-u^2/2} .$$

The Rayleigh distribution is very important in communication engineering and is also useful in reliability theory. It is a special case of a two parameter Weibull distribution. A survey of the literature and the relations between the Rayleigh, Weibull and extreme-value distributions can be found in Johnson and Kotz (1970).

Best linear unbiased estimates of the parameter based on censored samples and complete samples were considered by Dyer and Whisenand (1973).

We will use our unified method to derive the ABLE σ^* and the unique optimum spacing for the ABLE σ^* based on order statistics selected from complete, singly censored or doubly censored samples in this chapter.

In the case of the Rayleigh distribution, the following formulas will be used to obtain the ABLE σ^*:

$$f_i = f(u_i) = (1-\lambda_i)[-2\ln(1-\lambda_i)]^{1/2}, \quad \sum = \sum_{i=1}^{k+1},$$

$$f_i u_i = -2(1-\lambda_i)\ln(1-\lambda_i),$$

$$K_2 = 4\sum \frac{[(1-\lambda_i)\ln(1-\lambda_i)-(1-\lambda_{i-1})\ln(1-\lambda_{i-1})]^2}{(1-\lambda_{i-1})-(1-\lambda_i)}$$

$$b_i = \frac{2(1-\lambda_i)[-2\ln(1-\lambda_i)]^{1/2}}{K_2} \cdot \left[\frac{(1-\lambda_i)\ln(1-\lambda_i)-(1-\lambda_{i-1})\ln(1-\lambda_{i-1})}{(1-\lambda_i)-(1-\lambda_{i-1})} - \frac{(1-\lambda_{i+1})\ln(1-\lambda_{i+1})-(1-\lambda_i)\ln(1-\lambda_i)}{(1-\lambda_{i+1})-(1-\lambda_i)}\right],$$

$$\lambda_0 = 0, \ \lambda_{k+1} = 1, \quad f_0 = f_{k+1} = f_0 u_0 = f_{k+1}u_{k+1} = 0,$$

$$\sigma^* = \sum_{i=1}^{k} b_i \, x_{(n_i)},$$

$$A\,V(\sigma^*) = \frac{\sigma^2}{n}\frac{1}{K_2},$$

$ARE(\sigma^*) = \frac{1}{4} K_2$ = asymptotic efficiency relative to the Cramér-Rao lower bound.

In order to determine the optimum spacing, we need the following notation to prove the lemmas and theorems:

$$\theta = \sigma, \quad \theta^* = \sigma^*, \quad K = K_2, \; m_i = f_i u_i$$

The Lemma 3.1 is established through the fact that $d^2m/d\lambda^2 = -2/(1-\lambda) \neq 0$.

6.2 Complete Sample (cf. (2.1))

Theorem 6.1: There is one and only one optimum spacing $\{\lambda_i^C\}_R = (\lambda_1^C, \lambda_2^C, \ldots, \lambda_k^C)$ for the ABLE σ^* based on k order statistics selected from a complete sample.

Proof: Lemma 3.2 and Theorem 3.1 follow from the substitutions

$h^*(\lambda) = 1-\lambda$, (i.e. $y_i = (1-\lambda_{k-i+1})/(1-\lambda_{k-i})$,

$i = 0, 1, \ldots, k$), $H_1(y) = 1+y\ln y/[2(1-y)]$,

$H_2(y) = -\ln y/[2(1-y)]$.

Then the theorem follows from Lemmas 3.1, 3.2, 3.3 and Theorem 3.1.

Algorithm:

Solving the following system of equations

$$H_1(y_{i-1}) = H_2(y_i), \quad i = 1, 2, \ldots, k$$

with $y_o^C = 0$, we have the unique set of values $\{y_i^C\} = (y_1^C, \ldots, y_k^C)$. Then the unique set $\{\lambda_i^C\}_R$ will be found from the relation

$$\lambda_i^C = 1 - y_k^C \cdot y_{k-1}^C \cdot \ldots \cdot y_{k-i+1}^C, \quad i=1,2,\ldots,k.$$

Table VI lists all the values of y_i^C for $k = 1(1)20$.

6.3 Singly Censored Sample

(A) Right censored sample (cf. (2.3))

Theorem 6.2: There is one and only one optimum spacing $\{\lambda_i^R\}_R = (\lambda_1^R, \lambda_2^R, \ldots, \lambda_k^R)$ for the ABLE σ^* based on k order statistics selected from a right censored sample.

Proof: (i) If $\beta \geq \lambda_k^C$, then $\{\lambda_i^R\}_R = \{\lambda_i^C\}_R$, (ii) if $\beta < \lambda_k^C$, the theorem follows from Lemmas 3.1, 3.2, 3.4 and Theorem 3.2 by the substitutions:

$$h^*(\lambda) = 1-\lambda, \; H_1(y) = 1+y \ln y/[2(1-y)],$$

$$H_2(y) = -\ln y/[2(1-y)].$$

Algorithm:

(i) If $\beta \geq \lambda_k^C$, take $\{\lambda_i^R\}_R = \{\lambda_i^C\}_R$.

(ii) If $\beta < \lambda_k^C$, use numerical iteration to obtain y_2^C, y_3^C, ..., y_k^C starting from $y_1^C = .5$ and shifting y_1^C to the right or the left of .5 successively until the relation

$$y_1^C \, y_2^C \, \ldots \, y_k^C = 1-\beta$$

is satisfied. Then

$$\lambda_k^R = \beta, \; \lambda_i^R = 1-(1-\beta)(y_1^C \cdot y_2^C \cdot \ldots \cdot y_{k-i}^C)^{-1}, \; i=1,2,\ldots,(k-1).$$

(B) Left censored sample (cf. (2.4))

Theorem 6.3: There is one and only one optimum spacing $\{\lambda_i^L\}_R = (\lambda_1^L, \lambda_2^L, \ldots, \lambda_k^L)$ for the ABLE σ^* based on k order statistics selected from a left censored sample.

Proof: (i) If $\alpha \leq \lambda_1^C$, then $\{\lambda_i^L\}_R = \{\lambda_i^C\}_R$, (ii) if $\alpha > \lambda_1^C$, the theorem follows from Lemmas 3.1, 3.2, 3.5 and Theorem 3.3 by the substitutions

$$h^*(\lambda) = 1-\lambda \; , \; H_1(y) = 1+y \ln y/[2(1-y)],$$

$$H_2(y) = - \ln y/[2(1-y)].$$

Algorithm:

(i) If $\alpha \leq \lambda_1^C$, take $\{\lambda_i^L\}_R = \{\lambda_i^C\}_R$.

(ii) If $\alpha > \lambda_1^C$, solve the following system of equations

$$H_1(y_{i-1}) = H_2(y_i), \quad i = 1, 2, \ldots, (k-1)$$

with $y_o^C = 0$ to obtain the unique set $\{y_i^C\} = (y_1^C, y_2^C, \ldots, y_{k-1}^C)$ which is coincident with the complete sample case. We then obtain the unique optimum spacing $\{\lambda_i^L\}_R$ as follows:

$$\lambda_1^L = \alpha, \ \lambda_i^L = 1-(1-\alpha)y_{k-i+1}^C y_{k-i+2}^C \cdot \ldots \cdot y_{k-1}^C ,$$

$$i = 2, 3, \ldots, k.$$

The values of y_i^C in Table VI can be applied here.

6.4 Doubly Censored Sample (cf. (2.5))

Theorem 6.4: There is one and only one optimum spacing $\{\lambda_i^D\}_R = (\lambda_1^D, \lambda_2^D, \ldots, \lambda_k^D)$ for the ABLE σ^* based on k order statistics selected from a doubly censored sample.

Proof: (i) If $\alpha \leq \lambda_1^C$ and $\beta \geq \lambda_k^C$, then $\{\lambda_i^D\}_R = \{\lambda_i^C\}_R$, (ii) if $\alpha \leq \lambda_1^R$ and $\beta < \lambda_k^C$, then $\{\lambda_i^D\}_R = \{\lambda_i^R\}_R$, (iii) if $\alpha > \lambda_1^C$ and $\beta \geq \lambda_k^L$, then $\{\lambda_i^D\}_R = \{\lambda_i^L\}_R$.
(iv) If $\alpha > \lambda_1^R$ and $\beta < \lambda_k^C$, or (v) if $\alpha > \lambda_1^C$ and $\beta < \lambda_k^L$, the theorem follows from Lemmas 3.1, 3.2, 3.6 and Theorem 3.4 by the substitutions:

$h^*(\lambda)=1-\lambda$, $H_1(y)=1+y\ln y/[2(1-y)]$, $H_2(y)= -\ln y/[2(1-y)]$.

Algorithm:

(i) If $\alpha \leq \lambda_1^C$ and $\beta > \lambda_k^C$, take $\{\lambda_i^D\}_R = \{\lambda_i^C\}_R$;

(ii) If $\alpha \leq \lambda_1^R$ and $\beta > \lambda_k^C$, take $\{\lambda_i^D\}_R = \{\lambda_i^R\}_R$;

(iii) If $\alpha > \lambda_1^C$ and $\beta \geq \lambda_k^L$, take $\{\lambda_i^D\}_R = \{\lambda_i^L\}_R$;

(iv) If $\alpha > \lambda_1^R$ and $\beta < \lambda_k^C$ or $\alpha > \lambda_1^C$ and $\beta < \lambda_k^L$,

use numerical iteration to obtain y_2^C, y_3^C, ..., y_{k-1}^C starting from $y_1^C = .5$ and shifting y_1^C to the right or the left of .5 successively until the relation

$$y_1^C \cdot y_2^C \cdot \ldots \cdot y_{k-1}^C = (1-\beta)/(1-\alpha)$$

is satisfied. Then the unique optimum spacing $\{\lambda_i^D\}_R$ is obtained by

$$\lambda_1^D = \alpha, \quad \lambda_k^D = \beta, \quad \lambda_i^D = 1-(1-\alpha)\, y_{k-1}^C \cdot y_{k-2}^C \cdot \ldots \cdot y_{k-i+1}^C ,$$

or $\lambda_i^D = 1-(1-\beta)(y_1^C\, y_2^C \ldots y_{k-i}^C)^{-1}$, $i = 2,3, \ldots, (k-1)$.

6.5 Remarks

Table VI can be used in the same way as Table III which was illustrated by an example in 5.5.

A FORTRAN IV program was written for finding the values of y_i^C for any values of k.

CHAPTER 7

THE DETERMINATION OF THE OPTIMUM SPACING FOR THE ABLE OF THE PARAMETER BASED ON ORDER STATISTICS SELECTED FROM A MULTIPLY CENSORED SAMPLE

7.1 Introduction:

Let

$$X_{m_1} < X_{m_1+1} < \dots < X_{m_1+\ell_1} < X_{m_2} < X_{m_2+1} <$$

$$\dots < X_{m_2+\ell_2} < \dots < X_{m_J} < X_{m_J+1} <$$

$$\dots < X_{m_J+\ell_J} \tag{7.1}$$

where $m_i = [n\alpha_{i1}]+1$, $m_i+\ell_i = [n\alpha_{i2}]+1$, $i=1,2,\dots,J$, $0 \leq \alpha_{11} < \alpha_{12} < \alpha_{21} < \alpha_{22} < \dots < \alpha_{J1} < \alpha_{J2} \leq 1$, be a large multiply censored sample (cf. (2.6)) from a distribution with pdf $g(x) = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$ where μ and σ are location and scale parameters respectively.

The ABLE of one parameter, when the other one is known based on k order statistics selected from the sample (7.1) is as stated in 2.1.

The problem is how to determine the optimum spacing for the ABLE. i.e. to find the set of values $\{\lambda_i\}$ which will give the maximum value to the ARE of the estimate.

In this chapter we will state and prove an algorithm for finding the optimum spacing $\{\lambda_i^m\}_L$ for the ABLE μ^* of the location parameter from the logisitc distribution and then give the application of this approach to the other distributions.

7.2 The Algorithm for the Logistic Distribution:

Let a given multiply censored sample (7.1) be from the logistic distribution with pdf and cdf as stated in 4.1.

In order to obtain the optimum spacing $\{\lambda_i^m\}_L$ for the ABLE μ^* of the location parameter, when the scale parameter σ is known, based on k order statistics selected from the sample (7.1), we have to maximize K_1 with respect to $\lambda_1, \lambda_2, \ldots, \lambda_k$ over the domain $0 \leq \alpha_{11} < \alpha_{12} < \alpha_{21} < \alpha_{22} < \ldots < \alpha_{J1} < \alpha_{J2} \leq 1$.

Let

$$I(\underset{\sim}{\alpha}_1,\underset{\sim}{\alpha}_2) = \{(\lambda_1,\ldots,\lambda_k) | \lambda_i \varepsilon [\alpha_{11},\alpha_{12}] \cup [\alpha_{21},\alpha_{22}] \cup \ldots \cup [\alpha_{J1},\alpha_{J2}] \text{ and } \lambda_1 < \lambda_2 < \ldots < \lambda_k\},$$

with $0 \leq \alpha_{11} < \alpha_{12} < \alpha_{21} < \alpha_{22} < \ldots < \alpha_{J1} < \alpha_{J2} \leq 1$ be the uncensored portion of an interval [0,1] and let $\underset{\sim}{\lambda}^C$ denote the optimum spacing $\{\lambda_i^C\}_L$ for the complete sample case and $\underset{\sim}{\lambda}^m = \{\lambda_i^m\}_L$ for the multiply censored sample case.

Algorithm:

If $\underset{\sim}{\lambda}^c \in I(\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2)$, then $\underset{\sim}{\lambda}^m = \underset{\sim}{\lambda}^c$. If $\underset{\sim}{\lambda}^c \notin I(\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2)$, the optimum spacing $\underset{\sim}{\lambda}^m = \{\lambda_i^m\}_L$ must be one of the $\sum_{i=1}^{2J} \binom{k}{i}\binom{2J}{i}$ possible spacings ($\binom{k}{i} = 0$ if $i > k$) which can be systematically obtained in the following way:

Step 1:

For every $j = 1, 2, \ldots, 2J$, let each of the $\binom{2J}{j}$ subsets $\underset{\sim}{\lambda}_j = \{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_j}\}$ of $\underset{\sim}{\lambda}$ equal one of the $\binom{2J}{j}$ subsets $\underset{\sim}{\alpha}_j = \{\alpha_{\gamma_1 s_1}, \alpha_{\gamma_2 s_2}, \ldots, \alpha_{\gamma_j s_j}\}$ (in the sense that $\lambda_{i_1} = \alpha_{\gamma_1 s_1}$, $\lambda_{i_2} = \alpha_{\gamma_2 s_2}$, $\ldots$, $\lambda_{i_j} = \alpha_{\gamma_j s_j}$).

Step 2:

If $\alpha_{\gamma s} < \alpha_{\gamma' s'}$ and $\lambda_i < \lambda_{i'}$ are such that $\lambda_i = \alpha_{\gamma s}$, $\lambda_{i'} = \alpha_{\gamma' s'}$, then

$$\lambda_{i+t} = \frac{(i'-i-t)\alpha_{\gamma s} + t\alpha_{\gamma' s'}}{i'-i}, \quad t=1,2,\ldots,(i'-i-1).$$

Step 3:

If $\alpha_{\gamma s} > 0$ is the smallest number in $\underset{\sim}{\alpha}_j$ and $\lambda_i = \alpha_{\gamma s}$, then $\lambda_t = \dfrac{t\alpha_{\gamma s}}{i}$, $t = 1, 2, \ldots, (i-1)$.

Step 4:

If $\alpha_{\gamma s} < 1$ is the largest number in $\underset{\sim}{\alpha}_j$ and $\lambda_i = \alpha_{\gamma s}$, then $\lambda_t = \dfrac{(k-i+1-t)\alpha_{\gamma s}+t}{k-i+1}$, $t = 1, 2, \ldots, (k-i)$.

Step 5:

If $\alpha_{\gamma s} = 0$ or 1 and $\lambda_i = \alpha_{\gamma s}$, then λ_j is disregarded.

Verification of the algorithm:

Let u_i be the population quantile satisfying $\lambda_i = \int_{-\infty}^{u_i} f(t)dt$ and let $f_i = f(u_i)$.

It has been shown in Lemma 3.1 that if K_1^* is obtained from K_1 by removing one of the λ_i's and keeping other λ_i's as before, then

$$K_1 - K_1^* > 0, \tag{7.2}$$

and the optimum spacing λ^C must be an interior point (i.e. $0 < \lambda_1^C < \lambda_2^C < \ldots < \lambda_k^C < 1$) and the unique solution of the system of equations (Lemma 3.1 and Lemma 3.3):

$$\frac{f_{i+1}-f_i}{\lambda_{i+1}-\lambda_i} + \frac{f_i-f_{i-1}}{\lambda_i-\lambda_{i-1}} - 2\frac{df_i}{d\lambda_i} = 0, \quad i=1,2,\ldots,k, \tag{7.3}$$

For the logistic distribution, (7.3) can be written as

$$\lambda_{i+1} - \lambda_i = \lambda_i - \lambda_{i-1}, \quad i=1,2,\ldots,k. \tag{7.4}$$

To simplify the notation, from now on, let us demonstrate the verification of the algorithm through the special case: $k = 3$ and $J = 2$ with $I=I(\underset{\sim}{\alpha}_1,\underset{\sim}{\alpha}_2)=\{(\lambda_1,\ldots,\lambda_k)|$ $\lambda_i\varepsilon[0,\alpha_{12}] \cup [\alpha_{21},\alpha_{22}],\lambda_1<\lambda_2<\ldots<\lambda_k\}$, $0<\alpha_{12}<\alpha_{21}<\alpha_{22}<1$.

If $\underset{\sim}{\lambda}^c = (\lambda_1^c, \lambda_2^c, \lambda_3^c) \notin I$, then $\underset{\sim}{\lambda}^m = (\lambda_1^m, \lambda_2^m, \lambda_3^m)$ must be a boundary point of I, i.e., $\lambda_i^m = \alpha_{\gamma s}$, for some i, γ and s. For if $\underset{\sim}{\lambda}^m$ is an interior point, it must satisfy (7.3) and this contradicts the uniqueness of the solutions of (7.3). Suppose that $\lambda_2^m = \alpha_{12}$, say, and that no other λ_i^m is equal to any other $\alpha_{\gamma s}$. λ_1^m and λ_3^m can be obtained by maximizing $K_1(\lambda_1, \alpha_{12}, \lambda_3)$ which can be rewritten as

$$K_1^{(1)}+K_1^{(3)} = [(\lambda_1-\lambda_o)(1-\lambda_1-\lambda_o)^2 + (\alpha_{12}-\lambda_1)(1-\alpha_{12}-\lambda_1)^2]$$
$$+[(\lambda_3-\alpha_{12})(1-\lambda_3-\alpha_{12})^2 + (\lambda_4-\lambda_3)(1-\lambda_4-\lambda_3)^2],$$

with $0 = \lambda_o < \lambda_1 < \alpha_{12} < \lambda_3 < \alpha_{22} < \lambda_4 = 1$.

Following arguments similar to those in the proof of (7.2) and in the proof of Theorem 4.1 that $\underset{\sim}{\lambda}^c$ is the unique solution of (7.4), it can be shown that the values of λ_1^m and λ_3^m which maximizes $K_1^{(1)}$ and $K_1^{(3)}$, respectively, must be the unique solutions of

$$\alpha_{12}-\lambda_1 = \lambda_1-0 \quad \text{and} \quad 1-\lambda_3 = \lambda_3-\alpha_{12} \,. \tag{7.5}$$

So $\lambda_1^m = \alpha_{12}/2$ and $\lambda_3^m = (1+\alpha_{12})/2$ as given by Steps 3 and 4.

Similarly, if $\lambda_1^m = \alpha_{12}$ and $\lambda_3^m = \alpha_{22}$, say, then λ_2^m is the unique solution of

$$\alpha_{22} - \lambda_2 = \lambda_2 - \alpha_{12}$$

i.e.
$$\lambda_2^m = (\alpha_{12} + \alpha_{22})/2$$

as given by Step 2.

It is impossible that any $\lambda_i^m = 0$. For example, suppose $\lambda_1^m = 0$. Then by (7.2) $K_1(\lambda_1, \lambda_2^m, \lambda_3^m) > K_1(0, \lambda_2^m, \lambda_3^m)$ for any $0 < \lambda_1 < \lambda_2^m$ which contradicts that $\underset{\sim}{\lambda}^m$ is an optimum spacing. So Step 5 is justified.

Finally, one may have $\lambda_1^m = \alpha_{12}$, $\lambda_2^m = \alpha_{21}$ and $\lambda_3^m = \alpha_{22}$, which is one of the spacings to be considered as indicated in Step 1.

Now we will give an example to illustrate how to apply the algorithm.

Example: Suppose $k = 3$ and $J = 2$ with $I = I(\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2) = \{(\lambda_1, \ldots, \lambda_k) | \lambda_i \in [0, .40] \cup [.80, .95]\}$, so that $\underset{\sim}{\lambda}^c = (1/4,\ 2/4,\ 3/4) \notin I$. The number of possible spacings to be considered is at most

$$\binom{3}{1}\binom{4}{1} + \binom{3}{2}\binom{4}{2} + \binom{3}{3}\binom{4}{3} = 12 + 18 + 4 = 34.$$

(1) One λ_i equals $\alpha_{\gamma s}$: the 12 spacings can be obtained from Steps 3 and 4. After eliminating infeasible spacings and spacings with at least one λ_i equals 0 as indicated in Step 5, the eligible spacings are (.133, .267, .400), (.800, .867, .933) and (.400, .800, .900).

(2) Two λ_i's equal $\alpha_{\gamma s}$'s: the 18 spacings can be obtained from Steps 2, 3 and 4. After eliminating infeasible spacings and spacings with at least one λ_i equals 0 as indicated in Step 5, the eligible spacings are (.400, .800, .900), (.200, .400, .800), (.200, .400, .950), (.800, .875, .950) and (.400, .800, .950).

(3) Three λ_i's equal $\lambda_{\gamma s}$'s: the 4 spacings can be obtained from Step 1. After eliminating the spacings with $\lambda_1 = 0$ as indicated in Step 5, the only eligible spacing is (.400, .800, .950).

After comparing the values of K_1 for the nine eligible spacings, we find that the optimum spacing is (.200, .400, .800) with $K_1 = .304$, i.e., with ARE (μ^*) = .912. The corresponding coefficients are $\{a_i\}_L = \{.210, .474, .316\}$.

The 34 spacings are not all distinct. For example, by letting $\lambda_2^m = .800$ and $\lambda_3^m = .950$, Step 3 gives $\lambda_1^m = .400$. On the other hand, (3) also gives the same spacing.

The manual computation of the spacings by using the algorithm may be tedious when k and J are large. A FORTRAN IV program for computing an optimum spacing and the corresponding values of the ARE and coefficients a_i has been written. For the above example, the execution time on the PDP-10 is .33 seconds, for $k = 4$ and $J = 3$, the execution time is 1.03 seconds and for $k = 5$ and $J = 3$ the execution time is 1.68 seconds. This program can also be used for complete samples, singly censored samples, doubly censored samples and middle censored samples which are all special cases of multiply censored samples.

7.3 Application to Other Distributions

(7.2) and the fact that the optimum spacing is the unique solution of (7.3) hold for the ABLE of the location parameter of the normal (Higuchi 1954, Ogawa 1972), Cauchy when $k \neq 4m-1$ (Chan 1970, Balmer, Boulton and Sack 1972) and extreme-value (Chan and Kabir 1969) distributions when the scale parameter is known. They also hold for the ABLE of the scale parameter (f_i in (7.3) is replaced by $f_i u_i$) of the Weibull (of which the exponential distribution (Saleh and Ali 1966) is the special case when the shape parameter is 1) and Pareto (Kulldorff and Vänmann 1973, Chan and Cheng 1972) distributions when the location and shape parameters are known and the ABLE of the parameter of the Rayleigh distribution. The algorithm can be applied to these distributions when samples are multiply censored like (7.1). Although the possible spacings for most cases cannot be expressed analytically, due to the uniqueness of the solutions of equations of the form (7.3) with any boundary conditions $\lambda_i = \alpha_{\gamma s}$ for some i, γ and s (e.g., (7.5)), they can be easily obtained by a standard equation solving technique like the Newton-Raphson method.

CHAPTER 8

AN ASYMPTOTICALLY UNIFORMLY MOST POWERFUL TEST OF THE LOCATION PARAMETER BASED ON ORDER STATISTICS

8.1 Introduction

Given a large ordered sample of size n,

$$X_{(1)} < X_{(2)} < \ldots < X_{(n)} \tag{8.1}$$

from a distribution with the pdf $(1/\sigma)f[(x-\mu)/\sigma]$ depending on the location parameter μ and scale parameter σ only, we want to test the simple hypothesis

$$H_o: \mu = \mu_o, \quad \sigma = \sigma_o$$

against a composite alternative hypothesis

$$H_1: \mu > \mu_o, \quad \sigma = \sigma_o$$

or

$$H_2: \mu < \mu_o, \quad \sigma = \sigma_o$$

by using $k(<n)$ order statistics selected from (8.1).

An asymptotically uniformly most powerful test (UMP test) is found.

Eisenberger (1965 (a),(b), 1968 (a),(b)) has dealt with similar cases to test the location and scale parameters independently and jointly (simple hypothesis against simple alternative hypothesis) based on one, two, four, six and eight sample quantiles. When based on more than two quantiles, the tests use linear combinations of the quantiles. Under all circumstances, symmetrical sample quantiles from the normal distribution were considered.

We are dealing with the general case, i.e. to find an asymptotically UMP test of the location parameter by using the k selected sample quantiles without the assumptions that the quantiles are symmetrically located and from a specified distribution.

Ogawa (1951) proposed a Student's t-test for testing the location parameter by using the k selected sample quantiles.

8.2 The Asymptotically UMP Test

We want to test the simple null hypothesis

$$H_o: \quad \mu = \mu_o, \quad \sigma = \sigma_o \quad \text{(known)}$$

against the composite alternative hypothesis

$$H_1: \quad \mu > \mu_o, \quad \sigma = \sigma_o \text{ (known)}$$

based on k order statistics

$$X_{(n_1)} < X_{(n_2)} < \ldots < X_{(n_k)} \tag{8.2}$$

from (8.1), where $n_i = [n\lambda_i] + 1$.

From Chapter 2, we see that the joint distribution of $X_{(n_1)}, X_{(n_2)}, \ldots, X_{(n_k)}$ is

$$\ell(x_{(n_1)}, \ldots, x_{(n_k)}) = h(\lambda_1,\ldots,\lambda_k,\sigma,n)\exp -\{\frac{n}{2\sigma^2}\Omega\},$$

where

$$h(\lambda_1,\ldots,\lambda_k,\sigma,n) = (\frac{n}{2\pi\sigma^2})^{k/2} f_1 \cdots f_k\{\lambda_1(\lambda_2-\lambda_1)\cdot$$

$$\cdot\ldots\cdot(\lambda_k-\lambda_{k-1})(1-\lambda_k)\}^{-1/2},$$

$$\Omega = \sum_{i=1}^{k} \frac{\lambda_{i+1}-\lambda_{i-1}}{(\lambda_{i+1}-\lambda_i)(\lambda_i-\lambda_{i-1})} f_i^2(x_{(n_i)}-\mu-\sigma u_i)^2$$

$$-2\sum_{i=2}^{k} \frac{f_i f_{i-1}}{\lambda_i-\lambda_{i-1}} (x_{(n_i)}-\mu-\sigma u_i)(x_{(n_{i-1})}-\mu-\sigma u_{i-1}),$$

First, we specify a value of μ, say, $\mu = \mu_1 > \mu_0$ for the alternative hypothesis. The best critical region (BCR) is then obtained by the Neyman -Pearson Lemma:

$$\frac{L_o}{L_1} = \frac{\exp\{-\frac{n}{2\sigma_o^2}\Omega_o\}}{\exp\{-\frac{n}{2\sigma_o^2}\Omega_1\}} \le C \qquad (8.3)$$

where Ω_i, $i = 0, 1$, is the value of Ω under the hypotheses H_o and H_1 respectively.

(8.3) can be simplified to

$$\exp\{ \frac{n}{2\sigma_o^2} (\mu_1-\mu_o)[2(X-K_3\sigma_o)-K_1(\mu_o+\mu_1)] \} \geq C, \qquad (8.4)$$

where $$X = \sum_{i=1}^{k+1} \frac{(f_i-f_{i-1})(f_i x(n_i)-f_{i-1}x(n_{i-1}))}{\lambda_i - \lambda_{i-1}}$$

$$= \sum_{i=1}^{k} \left(\frac{f_i-f_{i-1}}{\lambda_i-\lambda_{i-1}} - \frac{f_{i+1}-f_i}{\lambda_{i+1}-\lambda_i}\right) f_i x(n_i) ,$$

$$K_1 = \sum_{i=1}^{k+1} \frac{(f_i-f_{i-1})^2}{\lambda_i-\lambda_{i-1}} , \quad K_3 = \sum_{i=1}^{k+1} \frac{(f_i-f_{i-1})(f_i u_i-f_{i-1}u_{i-1})}{\lambda_i - \lambda_{i-1}} .$$

Let $$M = \frac{\sqrt{nK_1}}{\sigma_o}(\mu_1-\mu_o), \quad T = \frac{\sqrt{n}}{\sigma_o\sqrt{K_1}} (X-K_3\sigma_o-K_1\mu_o) ,$$

(8.4) can then be expressed as

$$\exp \{M(T-M/2)\} \geq C$$

which is equivalent to

$$T \geq C'$$

since $M > 0$ and where $C' = (\ln C)/M + M/2$.

Asymptotically, X has the normal distribution with

$$E(X) = E[\sum_{i=1}^{k} \left(\frac{f_i-f_{i-1}}{\lambda_i-\lambda_{i-1}} - \frac{f_{i+1}-f_i}{\lambda_{i+1}-\lambda_i}\right) f_i x(n_i)]$$

$$= \sum_{i=1}^{k} \left(\frac{f_i-f_{i-1}}{\lambda_i-\lambda_{i-1}} - \frac{f_{i+1}-f_i}{\lambda_{i+1}-\lambda_i}\right) f_i(\mu+\sigma u_i)$$

$$= K_1\mu + K_3\sigma \ ,$$

$$\mathrm{Var}(X) = \mathrm{Var}\Big[\sum_{i=1}^{k} \left(\frac{f_i - f_{i-1}}{\lambda_i - \lambda_{i-1}} - \frac{f_{i+1} - f_i}{\lambda_{i+1} - \lambda_i} \right) f_i x_{(n_i)} \Big]$$

$$= \frac{\sigma^2}{n} K_1 \quad .$$

So T is asymptotically normally distributed with mean 0 and variance 1.

Theorem 8.1: The asymptotically UMP test for H_o: $\mu = \mu_o$, $\sigma = \sigma_o$ (known) against H_1: $\mu > \mu_o$, $\sigma = \sigma_o$ (known) at the level of significance α is

$$X \geq \frac{p_\alpha \sigma_o \sqrt{K_1}}{\sqrt{n}} + K_3\sigma_o + K_1\mu_o \qquad (8.5)$$

where p_α is such that

$$\alpha = \int_{p_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy \quad .$$

Proof: The asymptotically UMP test for H_o against H_1 is obtained by the Neyman -Pearson Lemma and therefore from the reduced form (8.4)

$$\exp\{ \frac{n}{2\sigma_o^2} (\mu-\mu_o)[2(X-K_3\sigma_o)-K_1(\mu+\mu_o)] \} \geq C \quad .$$

This can be simplified to

$$T \geq C' \qquad (8.6)$$

by the substitutions

$$M = \frac{\sqrt{nK_1}}{\sigma_o}(\mu-\mu_o), \quad T = \frac{\sqrt{n}}{\sigma_o\sqrt{K_1}}(X-K_3\sigma_o-K_1\mu_o) ,$$

and $C' = (\ln C)/M + M/2$.

Since $T \sim N(0,1)$ asymptotically, p_α is such that

$$\alpha = \int_{p_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy ,$$

and let $C = \exp\{Mp_\alpha - M^2/2\}$. Then (8.6) is equivalent to $T \geq p_\alpha$,

i.e. $$X \geq \frac{p_\alpha\sigma_o\sqrt{K_1}}{\sqrt{n}} + K_3\sigma_o + K_1\mu_o \quad .$$

By a similar proof, we have the following theorem.

Theorem 8.2: The asymptotically UMP test for H_o: $\mu = \mu_o$, $\sigma = \sigma_o$ (known) against H_2: $\mu < \mu_o$, $\sigma = \sigma_o$ (known) at the level of significance α is

$$X \leq K_3\sigma_o + K_1\mu_o - \frac{p_\alpha\sigma_o\sqrt{K_1}}{\sqrt{n}} , \tag{8.7}$$

where p_α is such that

$$\alpha = \int_{-\infty}^{-p_\alpha} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy .$$

8.3 The Power Function of the Test

(1) For the test H_o against H_1, i.e.

$$X \geq \frac{p_\alpha \sigma_o \sqrt{K_1}}{\sqrt{n}} + K_3\sigma_o + K_1\mu_o \quad ,$$

the power function is

$$P_1 = 1 - \Phi(z_1)$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$, and

$$z_1 = (K_3\sigma_o + K_1\mu_o - K_3\sigma_o - K_1\mu)\frac{\sqrt{n}}{\sigma_o\sqrt{K_1}} + p_\alpha$$

$$= \frac{\sqrt{nK_1}}{\sigma_o}(\mu_o - \mu) + p_\alpha$$

For fixed α, μ, k and n, it is obvious that P_1 is an increasing function of K_1. In order to obtain the maximum power of the test, we have to maximize K_1 with respect to $\lambda_1, \lambda_2, \ldots, \lambda_k$. Therefore the optimum spacing for the ABLE μ^* is the solution which will surely give the largest power of the test.

(2) For the test H_o against H_2, i.e.

$$X \leq K_3\sigma_o + K_1\mu_o - \frac{p_\alpha \sigma_o \sqrt{K_1}}{\sqrt{n}} \quad ,$$

the power function is

$$P_2 = \Phi(z_2)$$

where $z_2 = \frac{\sqrt{nK_1}}{\sigma_o} (\mu_o - \mu) - p_\alpha$.

We can see that the power function P_2 is an increasing function of K_1, similarly, the optimum spacing for the ABLE μ^* will provide the largest power of the test.

8.4 Comparison of the Power With Ogawa's t-Test

We did a few comparisons of the Power between Ogawa's t-test and our asymptotically UMP test:

H_o: $\mu_o=0$, $\sigma_o=1$, H_1: $\mu=1>\mu_o$, $\sigma_o=1$, α = 1%, 5%, 10%

k = 3, 4, 5.

Under all circumstances as above, Ogawa's tests have power less than 15% and at the same time, our tests have power larger than 28% even the sample size n is as small as 10.

We could conclude that our test is better than Ogawa's t-test if we want to use one sided test for testing the location parameter.

8.5 Remarks

From Theorems 8.1 and 8.2, we can no longer obtain the same BCR for H_o against the composite alternative hypothesis $H_3: \mu \neq \mu_o$ which includes values of μ both to the right and to the left of the point μ_o. It therefore follows that there is no asymptotically UMP test existing for H_o against H_3.

The tests for the scale parameter or both location and scale parameters are still open.

TABLE I

The optimum spacings $\{\lambda_i\}_L$ and the corresponding coefficients of the ABLE $\mu^* = \sum_{i=1}^{k} a_i X_{([n\lambda_i])+1} - \sigma \frac{K_3}{K_1}$

based on complete and censored samples from the logistic distribution

Samples	Censoring Proportions		λ_i	a_i
	Left α	Right $1-\beta$		
Complete	0	0	$\frac{i}{k+1}$	$\frac{2i}{K^2}(1-\frac{i}{k+1})$
Right	0	$\beta \geq \frac{k}{k+1}$	$\frac{i}{k+1}$	$\frac{2i}{k^2}(1-\frac{i}{k+1})$
Censored	0	$\beta < \frac{k}{k+1}$	$\frac{i}{k}\beta$	$\frac{2i}{k^2}(1-\frac{i}{k+1})$
Left	$\alpha \leq \frac{1}{k+1}$	0	$\frac{i}{k+1}$	$\frac{2i}{k^2}(1-\frac{i}{k+1})$
Censored	$\alpha < \frac{1}{k+1}$	0	$\frac{i-1}{k}(1-\alpha)+\alpha$	$\frac{2\{(i-1)(1-\alpha)+k\alpha\}\{k-i+i\alpha\}}{k^2(k+1)(1-\alpha)^2}$
Doubly	$\alpha \leq \frac{1}{k+1}$	$\beta \geq \frac{k}{k+1}$	$\frac{i}{k+1}$	$\frac{2i}{k^2}(1-\frac{i}{k+1})$
Censored	$\alpha \leq \frac{\beta}{k}$	$\beta < \frac{k}{k+1}$	$\frac{i}{k}\beta$	$\frac{2i}{k^2}(1-\frac{i}{k+1})$
	$\alpha > \frac{1}{k+1}$	$\beta \geq \frac{k-1+\alpha}{k}$	$\frac{i-1}{k}(1-\alpha)+\alpha$	$\frac{2\{(i-1)(1-\alpha)+k\alpha\}\{k-i+i\alpha\}}{k^2(k+1)(1-\alpha)^2}$
	$\alpha > \frac{\beta}{k}$	$\beta < \frac{k}{k+1}$	$\frac{(k-i)\alpha+(i-1)\beta}{k-1}$	$\frac{\{2(k-i)\alpha+(i-1)\beta\}\{2(i-1)\alpha+(k-i)\beta\}}{k^2(k+1)(\beta-\alpha)^2}$
	$\alpha > \frac{1}{k+1}$	$\beta < \frac{k-1+\alpha}{k}$	$\frac{(k-i)\alpha+(i-1)\beta}{k-1}$	$\frac{\{2(k-i)\alpha+(i-1)\beta\}\{2(i-1)\alpha+(k-i)\beta\}}{k^2(k+1)(\beta-\alpha)^2}$

TABLE II

The expressions for ARE(μ^*) and K_3 for the ABLE μ^* of the logistic distribution

Samples	Censoring Proportions Left α	Censoring Proportions Right $1-\beta$	ARE(μ^*)	K_3
Complete	0	0	$\frac{k(k+2)}{(k+1)^2}$	$\frac{2}{k+1}\cdot\sum_{i=1}^{k} g_i$
Right	0	$\beta \geq \frac{k}{k+1}$	$\frac{k(k+2)}{(k+1)^2}$	$\frac{2}{k+1}\cdot\sum_{i=1}^{k} g_i$
Censored	0	$\beta < \frac{k}{k+1}$	$3(\frac{k^2-1}{3k^2}\cdot\beta^3-\beta^2+\beta)$	$\frac{2\beta}{k}\cdot\sum_{i=1}^{k-1} g_i+(1-\frac{k-1}{k}\cdot\beta)\cdot g_k$
Left	$\alpha \leq \frac{1}{k+1}$	0	$\frac{k(k+2)}{(k+1)^2}$	$\frac{2}{k+1}\cdot\sum_{i=1}^{k} g_i$
Censored	$\alpha > \frac{1}{k+1}$	0	$(1-\alpha^3)-\frac{1}{k^2}(1-\alpha)^3$	$\frac{2(1-\alpha)}{k}\cdot\sum_{i=2}^{k} g_i+(\frac{1}{k}+\frac{k-1}{k}\cdot\alpha)\cdot g_1$
	$\alpha \leq \frac{1}{k+1}$	$\beta \geq \frac{k}{k+1}$	$\frac{k(k+2)}{(k+1)^2}$	$\frac{2}{k+1}\cdot\sum_{i=1}^{k} g_i$
Doubly	$\alpha \leq \frac{\beta}{k}$	$\beta < \frac{k}{k+1}$	$3(\frac{k^2-1}{3k^2}\cdot\beta^3-\beta^2+\beta)$	$\frac{2\beta}{k}\cdot\sum_{i=1}^{k-1} g_i+(1-\frac{k-1}{k}\cdot\beta)\cdot g_k$
Censored	$\alpha > \frac{1}{k+1}$	$\beta \geq \frac{k-1+\alpha}{k}$	$(1-\alpha^3)-\frac{1}{k^2}\cdot(1-\alpha)^3$	$\frac{2(1-\alpha)}{k}\cdot\sum_{i=2}^{k} g_i+(\frac{1}{k}+\frac{k-1}{k}\cdot\alpha)\cdot g_1$
	$\alpha > \frac{\beta}{k}$	$\beta < \frac{k}{k+1}$	$\frac{k(k-2)(\beta^3-\alpha^3)+3\alpha\beta(\beta-\alpha)}{(k-1)^2}-3\beta^2+3\beta$	$\frac{2(\beta-\alpha)}{k-1}\cdot\sum_{i=2}^{k-1} g_i+(\frac{k-2}{k-1}\alpha+\frac{1}{k-1}\cdot\beta)\cdot g_1+(1-\frac{1}{k-1}\alpha-\frac{k-2}{k-1}\cdot\beta)\cdot g_k$
	$\alpha > \frac{1}{k+1}$	$\beta < \frac{k-1+\alpha}{k}$	$\frac{k(k-2)(\beta^3-\alpha^3)+3\alpha\beta(\beta-\alpha)}{(k-1)^2}-3\beta^2+3\beta$	$\frac{2(\beta-\alpha)}{k-1}\cdot\sum_{i=2}^{k-1} g_i+(\frac{k-2}{k-1}\alpha+\frac{1}{k-1}\beta)g_1+(1-\frac{1}{k-1}\alpha-\frac{k-2}{k-1}\beta)g_k$

$g_i = \lambda_i(1-\lambda_i)[\ln\lambda_i - \ln(1-\lambda_i)]$ $i = 1, 2, \ldots, k$

$-\frac{K_3}{K_1} = -\frac{3K_3}{ARE(\mu^*)}$ = the coefficient of σ in μ^*

TABLE III

The values of y_i^C, $i = 1,2,\ldots,k$ for finding $\{\lambda_i^C\}_P$, $\{\lambda_i^L\}_P$

for the ABLE σ^* of the Pareto distribution

k \ γ	0.5	1.0	1.5	2.0	2.5	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
1	.410097	.500000	.566750	.618034	.658584	.691414	.741271	.777307	.804554	.825871	.843002	.857067	.868822
2	.602121	.666667	.713339	.748606	.776173	.798306	.831626	.855509	.873463	.887450	.898655	.907831	.915484
3	.700652	.750000	.785447	.812116	.832900	.849550	.874558	.892441	.905863	.916309	.924668	.931509	.937211
4	.760229	.800000	.828494	.849897	.866557	.879892	.899903	.914201	.924926	.933268	.939941	.945402	.949952
5	.800077	.833333	.857130	.874991	.888888	.900006	.916683	.928594	.937526	.944472	.950029	.954574	.958362
6	.828587	.857143	.877563	.892883	.904799	.914331	.928626	.938834	.946488	.952439	.957200	.961094	.964339
7	.849989	.875000	.892878	.906288	.916717	.925058	.937566	.946497	.953193	.958400	.962564	.965971	.968809
8	.866644	.888889	.904786	.916708	.925978	.933393	.944511	.952450	.958401	.963028	.966730	.969757	.972280
9	.879973	.900000	.914310	.925040	.933384	.940058	.950064	.957208	.962564	.966728	.970059	.972784	.975054
10	.890881	.909091	.922101	.931856	.939442	.945508	.954605	.961099	.965968	.969754	.972782	.975258	.977322
11	.899973	.916666	.928593	.937535	.944488	.950049	.958387	.964340	.968803	.972274	.975049	.977320	.979212
12	.907666	.923077	.934085	.942340	.948758	.953891	.961588	.967083	.971202	.974406	.976968	.979064	.980810
13	.914261	.928571	.938793	.946458	.952418	.957184	.964331	.969433	.973258	.976233	.978612	.980558	.982180
14	.919977	.933333	.942873	.950027	.955589	.960037	.966707	.971470	.975040	.977816	.980037	.981853	.983367
15	.924979	.937500	.946443	.953150	.958364	.962534	.968787	.973252	.976599	.979202	.981284	.982986	.984405
16	.929392	.941176	.949593	.955905	.960812	.964737	.970622	.974824	.977975	.980424	.982384	.983986	.985322
17	.933315	.944445	.952393	.958354	.962989	.966695	.972253	.976222	.979197	.981511	.983361	.984875	.986136
18	.936825	.947368	.954898	.960546	.964936	.968447	.973713	.977472	.980291	.982483	.984236	.985670	.986865
19	.939984	.950000	.957153	.962518	.966688	.970024	.975026	.978598	.981276	.983358	.985023	.986385	.987521
20	.942842	.952381	.959193	.964302	.968274	.971451	.976215	.979616	.982167	.984150	.985736	.987033	.988115

TABLE IV

The optimum spacings $\{\lambda_i^R\}_p$, coefficients b_i and ARE(σ^*)

of the ABLE $\sigma^*=\sum_{i=1}^{k} b_i X_{([n\lambda_i]+1)} - \sum_{i=1}^{k} b_i\mu$ based on right

censored samples with censoring proportions 1-β from the Pareto distribution

γ	k	β	λ_1	λ_2	λ_3	λ_4	λ_5	b_1	b_2	b_3	b_4	b_5	ARE
		.7	.3834	.7000				1.5142	.7094				.8571
	2	.6	.3223	.6000				1.5085	1.1635				.8099
		.5	.2642	.5000				1.5049	1.8119				.7340
		.8	.3098	.5789	.8000			1.3044	.6747	.2988			.9201
	3	.7	.2627	.4973	.7000			1.2419	.7451	.5932			.8864
		.6	.2196	.4202	.6000			1.1909	.8012	1.0184			.8266
2		.8	.2361	.4500	.6393	.8000		1.1008	.7102	.4047	.2628		.9375
	4	.7	.1996	.3838	.5512	.7000		1.0281	.7254	.4753	.5403		.8967
		.6	.1664	.3224	.4672	.6000		.9688	.7349	.5333	.9508		.8325
		.8	.1906	.3674	.5290	.6739	.8000	.9437	.6794	.4584	.2807	.2423	.9456
	5	.7	.1609	.3120	.4529	.5825	.7000	.8716	.6688	.4927	.3435	.5099	.9015
		.6	.1339	.2613	.3817	.4948	.6000	.8130	.6576	.5187	.3962	.9114	.8352
		.8	.4665	.8000				2.0376	.7400				.8581
	2	.7	.3949	.7000				2.0216	1.2788				.8241
		.6	.3298	.6000				2.0127	1.9954				.7601
		.8	.3258	.5936	.8000			1.6907	.9631	.5993			.8987
	3	.7	.2732	.5071	.7000			1.6120	1.0468	1.0949			.8470
		.6	.2264	.4267	.6000			1.5505	1.1113	1.7776			.7727
3		.9	.3003	.5522	.7533	.9000		1.5340	.9040	.4813	.2020		.9448
	4	.8	.2498	.4674	.6513	.8000		1.4078	.9621	.6008	.5364		.9131
		.7	.2086	.3953	.5593	.7000		1.3177	.9772	.6874	1.0102		.8551
		.9	.2443	.4577	.6393	.7874	.9000	1.3222	.9143	.5812	.3229	.1823	.9569
	5	.8	.2025	.3844	.5452	.6840	.8000	1.1977	.8982	.6417	.4281	.5006	.9197
		.7	.1686	.3233	.4637	.5894	.7000	1.1091	.8823	.6814	.5064	.9613	.8589
		.8	.4752	.8000				2.5484	1.1190				.8454
	2	.7	.4007	.7000				2.5276	1.8647				.7998
		.6	.3336	.6000				2.5162	2.8407				.7277
		.9	.4023	.7023	.9000			2.2171	1.1031	.3922			.9135
	3	.8	.3341	.6007	.8000			2.0792	1.2461	.9206			.8814
		.7	.2786	.5119	.7000			1.9854	1.3430	1.6154			.8197
4		.9	.3117	.5659	.7619	.9000		1.8729	1.1813	.6475	.3381		.9374
	4	.8	.2571	.4761	.6571	.8000		1.7183	1.2116	.7929	.8313		.8941
		.7	.2133	.4010	.5633	.7000		1.6119	1.2272	.8947	1.4999		.8267
		.9	.2544	.4716	.6516	.7944	.9000	1.6074	1.1391	.7509	.4429	.3076	.9485
	5	.8	.2088	.3931	.5531	.6888	.8000	1.4557	1.1164	.8220	.5724	.7801	.9000
		.7	.1727	.3290	.4690	.5927	.7000	1.3514	1.0960	.8673	.6653	1.4329	.8299
		.8	.4804	.8000				3.0584	1.5066				.8348
	2	.7	.4041	.7000				3.0332	2.4575				.7821
		.6	.3359	.6000				3.0194	3.6915				.7054
		.9	.4104	.7087	.9000			2.6320	1.3614	.5568			.9084
	3	.8	.3391	.6050	.8000			2.4689	1.5266	1.2512			.8680
		.7	.2819	.5148	.7000			2.3604	1.6368	2.1438			.8001
5		.9	.3191	.5741	.7669	.9000		2.2124	1.4274	.8127	.4835		.9309
	4	.8	.2615	.4813	.6604	.8000		2.0306	1.4600	.9829	1.1357		.8798
		.7	.2161	.4045	.5656	.7000		1.9082	1.4764	1.0998	1.9979		.8065
		.9	.2607	.4800	.6589	.7984	.9000	1.8934	1.3632	.9196	.5625	.4421	.9414
	5	.8	.2126	.3984	.5579	.6915	.8000	1.7156	1.3344	1.0009	.7152	1.0694	.8852
		.7	.1752	.3325	.4722	.5946	.7000	1.5959	1.3098	1.0519	.8223	1.9133	.8094

TABLE V

The optimum spacings $\{\lambda_i^D\}_P$, coefficients b_i and ARE(σ^*) of the ABLE $\sigma^* = \sum_{i=1}^{k} b_i X_{([n\lambda_i])+1)} - \sum_{i=1}^{k} b_i \mu$ based on doubly censored sample with the lower censoring proportions α and the upper censoring proportions $1-\beta$ from the Pareto distribution

γ	k	α	β	λ_1	λ_2	λ_3	λ_4	λ_5	b_1	b_2	b_3	b_4	b_5	ARE
2	2	.40	.70	.4000	.7000				1.4609	.6962				.8567
		.45	.75	.4500	.7500				1.4035	.5110				.8668
		.35	.60	.3500	.6000				1.4281	1.1301				.8091
	3	.35	.80	.3500	.5961	.8000			1.2477	.5948	.2905			.9189
		.30	.70	.3000	.5138	.7000			1.2036	.6596	.5797			.8856
		.25	.60	.2500	.4340	.6000			1.1710	.7198	1.0012			.8262
	4	.30	.80	.3000	.4884	.6560	.8000		1.0757	.5846	.3443	.2545		.9356
		.25	.70	.2500	.4147	.5652	.7000		1.0270	.6182	.4159	.5286		.8957
		.35	.65	.3500	.4567	.5569	.6500		.9880	.3942	.3004	.6824		.8556
	5	.25	.75	.2500	.3911	.5221	.6421	.7500	.9333	.5378	.3894	.2648	.3524	.9246
		.20	.70	.2000	.3396	.4699	.5903	.7000	.8956	.5937	.4428	.3140	.5032	.9011
		.25	.65	.2500	.3592	.4625	.5596	.6500	.9062	.4708	.3724	.2855	.6696	.8680
3	2	.50	.80	.5000	.8000				1.8995	.7131				.8566
		.45	.75	.4500	.7500				1.9495	.9705				.8450
		.40	.70	.4000	.7000				2.0028	1.2722				.8240
	3	.35	.80	.3500	.6033	.8000			1.6573	.8997	.5906			.8984
		.30	.70	.3000	.5184	.7000			1.5877	.9664	1.0794			.8467
		.30	.60	.3000	.4592	.6000			1.5131	.8608	1.7166			.7707
	4	.30	.85	.3000	.5225	.7065	.8500		1.4548	.8867	.5161	.3493		.9326
		.30	.80	.3000	.4962	.6633	.8000		1.3993	.8399	.5366	.5253		.9122
		.25	.75	.2500	.4428	.6098	.7500		1.3619	.9153	.6160	.7472		.8867
	5	.25	.85	.2500	.4374	.6005	.7385	.8500	1.2646	.8452	.5777	.3607	.3221	.9415
		.25	.75	.2500	.3970	.5297	.6475	.7500	1.1975	.7466	.5665	.4112	.6972	.8908
		.30	.65	.3000	.3974	.4883	.5726	.6500	1.2114	.5486	.4575	.3746	1.2148	.8162
4	2	.50	.80	.5000	.8000				2.4304	1.0904				.8446
		.45	.75	.4500	.7500				2.4774	1.4500				.8264
		.45	.70	.4500	.7000				2.3259	1.7798				.7976
	3	.35	.80	.3500	.6069	.8000			2.0567	1.1945	.9126			.8813
		.30	.70	.3000	.5208	.7000			1.9674	1.2637	1.5985			.8195
		.35	.65	.3500	.5115	.6500			1.8826	1.0095	1.9590			.7778
	4	.30	.85	.3000	.5279	.7112	.8500		1.7828	1.1491	.7026	.5614		.9194
		.35	.80	.3500	.5285	.6785	.8000		1.7099	.9474	.6459	.8030		.8912
		.30	.70	.3000	.4518	.5852	.7000		1.6501	.9591	.7257	1.4579		.8247
	5	.25	.85	.2500	.4428	.6071	.7428	.8500	1.5356	1.0759	.7586	.4965	.5205	.9275
		.25	.75	.2500	.4001	.5334	.6501	.7500	1.4620	.9493	.7382	.5536	1.0633	.8665
		.20	.70	.2000	.3475	.4800	.5975	.7000	1.3890	1.0214	.8140	.6300	1.4229	.8298
5	2	.50	.80	.5000	.8000				2.9527	1.4772				.8344
		.45	.75	.4500	.7500				2.9985	1.9379				.8121
		.45	.70	.4500	.7000				2.8209	2.3570				.7803
	3	.40	.85	.4000	.6646	.8500			2.4885	1.3548	.8700			.8924
		.35	.80	.3500	.6091	.8000			2.4530	1.4847	1.2441			.8680
		.30	.65	.3000	.4910	.6500			2.2938	1.4749	2.6414			.7588
	4	.35	.90	.3500	.5902	.7728	.9000		2.1961	1.3279	.7652	.4771		.9305
		.30	.80	.3000	.5027	.6690	.8000		2.0368	1.3288	.9085	1.1201		.8794
		.25	.75	.2500	.4485	.6149	.7500		1.9710	1.4256	1.0195	1.5296		.8452
	5	.30	.90	.3000	.5045	.6720	.8034	.9000	1.9119	1.2498	.8503	.5271	.4363	.9410
		.25	.75	.2500	.4019	.5357	.6516	.7500	1.7272	1.1496	.9076	.6942	1.4389	.8487
		.30	.70	.3000	.4168	.5223	.6166	.7000	1.7765	.9321	.7737	.6301	1.8591	.8065

TABLE VI

The values of y_i^C, $i = 1,2,\ldots,k$ for finding $\{\lambda_i^C\}_R$, $\{\lambda_i^L\}_R$ for the ABLE σ^* of the Rayleigh distribution

i	y_i^C	i	y_i^C
1	.203187	11	.779404
2	.361469	12	.794478
3	.470465	13	.807628
4	.548575	14	.819199
5	.606943	15	.829460
6	.652096	16	.838621
7	.688020	17	.846848
8	.717262	18	.854279
9	.741516	19	.861022
10	.761952	20	.867170

APPENDIX I

PROGRAM FOR FINDING VALUES OF y_i^C OF THE PARETO DISTRIBUTION

```
C       TO FIND THE VALUES OF X(I)'S FOR THE GIVEN VALUE
C       OF SHAPE PARAMETER FROM 'PARETO DIST.' (SCALE)
        DIMENSION X(101),B(101)
        DO 100 II=1,10
        READ 1,K,R
1       FORMAT(I,F)
        X(1)=.0
        B(1)=2.0/R
        DO 10 I=1,K
        CALL NEWTON(X(I),B(I),R,X(I+1))
        IF(X(I+1).EQ.1.0) GO TO 100
        B(I+1)=(X(I+1)**R*(X(I+1)-1.0))/(1.0-X(I+1)**R)+2.0/R
10      CONTINUE
        PRINT 7,R
7       FORMAT(// 4X,'PARETO DIST. WITH SHAPE PARAMETER =',
     CF5.2 )
        PRINT 2
2       FORMAT(// 4X,'X(K),X(K-1),...,X(0) ARE' /)
        PRINT 3,(X(I),I=1,K+1)
3       FORMAT(1X,5F12.8 /)
100     CONTINUE
        STOP
        END
        SUBROUTINE NEWTON(XO,B,R,Z)
        DIMENSION C(3000),F(3000),Y(3000)
        C(1)=100.0
        I=1
        Y(1)=XO
50      Y(I+1)=(B*R*Y(I)**(R+1.0)-1.0)/(B*(R+1.0)*Y(I)**R-
     C(B+1.0))
        D=.0000005
        C(I+1)=ABS(Y(I+1)-Y(I))
        IF(C(I+1).GT.D) GO TO 10
        F(I+1)=B*Y(I+1)**(R+1.0)-(B+1.0)*Y(I+1)+1.0
        IF(F(I+1).LE.D) GO TO 101
10      IF(C(I+1).LE.C(I)) GO TO 20
        Y(I)=XO+.05
        GO TO 50
20      IF(I.GE.1000) GO TO 30
        I=I+1
        GO TO 50
30      PRINT 33,I
33      FORMAT(1X,'I=',I4 /)
        Z=.0
        GO TO 44
101     Z=Y(I+1)
44      RETURN
        END
```

PROGRAM FOR FINDING $\{\lambda_i^R\}_p$, CORRESPONDING b_i's, ARE(σ*)'s OF THE ABLE σ* BASED ON RIGHT CENSORED SAMPLES FROM THE PARETO DISTRIBUTION

```
C       FIND O. SP. OF "PARETO" FOR RIGHT-CENSORING CASE
        DIMENSION Y(16),W(16)
        DIMENSION X(52),OS(16)
        DIMENSION A(20),B(20)
        ACCEPT 1,K,R,CL,CR
1       FORMAT(I,3F)
C       K=NO. OF O. S.
C       R=SHAPE
C       CL=L.-CENSORING PT.    CR=R.-CENSORING PT.
C
        DO 111 LL=1,10
        CR=CR-.025
        PRINT 80,R,CR
80      FORMAT(1X,'SHAPE=',F5.2,4X,'CR=',F8.6)
        X(1)=.0
        X(2)=.5
        DO 22 J=2,50
        Y(1)=X(J)
        DO 10 I=1,K-1
        W(I)=P(Y(I),R)
        CALL NEWTON(W(I),R,Y(I),Y(I+1))
10      CONTINUE
        PP=1.0
        DO 11 I=1,K
11      PP=PP*Y(I)
        Q=(1.0-CR)**(1.0/R)
        D=PP-Q
        IF(ABS(D).LE..0000002) GO TO 12
        IF(D) 21,12,23
21      X(J+1)=X(J)+ABS(X(J)-X(J-1))/2.0
        GO TO 22
23      X(J+1)=X(J)-ABS(X(J)-X(J-1))/2.0
22      CONTINUE
12      U=1.0
        DO 33 I=1,K
        U=U*Y(K+1-I)
33      OS(I)=1.0-U**R
        PRINT 4
4       FORMAT(/ 2X,'THE OPTIMUM SPACING IS:' /)
        PRINT 2,(OS(I),I=1,K)
2       FORMAT(2X,F10.8 /)
        DO 50 L=2,K
50      A(L)=((1.0-OS(L))**(1.0/R)-(1.0-OS(L-1))**(1.0/R))/(1.0
     C/(1.0-OS(L))-1.0/(1.0-OS(L-1)))
        A(1)=((1.0-OS(1))**(1.0/R)-1.0)/(1.0/(1.0-OS(1))-1.0)
        VK=A(1)*((1.0-OS(1))**(1.0/R)-1.0)
        DO 60 L=2,K
60      VK=VK+A(L)*((1.0-OS(L))**(1.0/R)-(1.0-OS(L-1))**(1.0/R))
        DO 70 L=1,K-1
70      B(L)=(A(L+1)-A(L))*(1.0-OS(L))**(1.0/R)/VK
        B(K)=(1.0-OS(K))**(1.0/R)*(-A(K))/VK
        B(K+1)=1.0+A(1)/VK
        RAE=R*(R+2.0)*VK
```

```
        PRINT 7
7       FORMAT(/ 2X,'THE COEFFICIENTS ARE:' /)
        PRINT 71,(B(I),I=1,K+1)
71      FORMAT(5F12.8 /)
        PRINT 72,VK,RAE
72      FORMAT(/ 2X,'K=',F12.8,5X,'RAE=',F12.8 ///)
111     CONTINUE
        STOP
        END
        FUNCTION P(X,T)
        P=T*(X**(T+1.0)-X**T)/2.0/(1.0-X**T)+.5
        RETURN
        END
        FUNCTION D(X,T)
        D=(T*T*X**T*(1.0-X)+T*X**T-T)/2.0/(X-X**(T+1.0))**2
        RETURN
        END
        FUNCTION F(Z,T,W)
        F=-.5*(1.0+T*(Z-1.0)/(Z-Z**(T+1.0)))-W
        RETURN
        END
        SUBROUTINE NEWTON(W,C,TT,Y)
        X0=.05
        I=1
10      P1=F(X0,C,W)
        F1=D(X0,C)
        XN=X0-P1/F1
        IF(XN.LE..0) GO TO 400
        DEL=ABS(XN-X0)/.0000002
        P2=F(XN,C,W)
        COF=ABS(P2)/.0000002
        IF(DEL.GT.1.0) GO TO 100
        IF(COF.GT.1.0) GO TO 100
        Y=XN
        GO TO 200
100     IF(I.GT.100) GO TO 400
        I=I+1
        X0=XN
        GO TO 10
400     PRINT 3,I
3       FORMAT(/ 2X,'I=',I3 /)
200     RETURN
        END
```

APPENDIX III

PROGRAM FOR FINDING $\{\lambda_i^D\}_p$, CORRESPONDING b_i's, ARE(σ^*)'S OF THE ABLE σ^* BASED ON DOUBLY CENSORED SAMPLES FROM THE PARETO DISTRIBUTION

```
C       FIND O. SP. OF PARETO FOR DOUBLE -CENSORING CASE
        DIMENSION Y(16),W(16)
        DIMENSION X(52),OS(16)
        DIMENSION A(20),B(20)
        DO 111 LL=1,10
        ACCEPT 1,K,R,CL,CR
 1      FORMAT(I,3F)
C       K=NO. OF O. S.
C       R=SHAPE
C       CL=L.-CENSORING PT.    CR=R.-CENSORING PT.
C
        PRINT 80,R,CL,CR
 80     FORMAT(1X,'SHAPE=',F5.2,4X,'CL=',F8.6,4X,'CR=',F8.6)
        X(1)=.0
        X(2)=.5
        DO 22 J=2,50
        Y(1)=X(J)
        DO 10 I=1,K-2
        W(I)=P(Y(I),R)
        CALL NEWTON(W(I),R,Y(I),Y(I+1))
 10     CONTINUE
        PP=1.0
        DO 11 I=1,K-1
 11     PP=PP*Y(I)
 41     Q=((1.0-CR)/(1.0-CL))**(1.0/R)
 44     D=PP-Q
        IF(ABS(D).LE..0000002) GO TO 12
        IF(D) 21,12,23
 21     X(J+1)=X(J)+ABS(X(J)-X(J-1))/2.0
        GO TO 22
 23     X(J+1)=X(J)-ABS(X(J)-X(J-1))/2.0
 22     CONTINUE
 12     U=1.0
        DO 33 I=2,K
        U=U*Y(K+1-I)
 33     OS(I)=1.0-U**R*(1.0-CL)
        OS(1)=CL
        PRINT 4
 4      FORMAT(/ 2X,'THE OPTIMUM SPACING IS:' /)
        PRINT 2,(OS(I),I=1,K)
 2      FORMAT(2X,F10.8 /)
        DO 50 L=2,K
 50     A(L)=((1.0-OS(L))**(1.0/R)-(1.0-OS(L-1))**(1.0/R))/(1.0
       C/(1.0-OS(L))-1.0/(1.0-OS(L-1)))
        A(1)=((1.0-OS(1))**(1.0/R)-1.0)/(1.0/(1.0-OS(1))-1.0)
        VK=A(1)*((1.0-OS(1))**(1.0/R)-1.0)
        DO 60 L=2,K
 60     VK=VK+A(L)*((1.0-OS(L))**(1.0/R)-(1.0-OS(L-1))**(1.0/R))
        DO 70 L=1,K-1
 70     B(L)=(A(L+1)-A(L))*(1.0-OS(L))**(1.0/R)/VK
        B(K)=(1.0-OS(K))**(1.0/R)*(-A(K))/VK
        B(K+1)=1.0+A(1)/VK
        RAE=R*(R+2.0)*VK
```

```
         PRINT 7
7        FORMAT(/ 2X,'THE COEFFICIENTS ARE:' /)
         PRINT 71,(B(I),I=1,K+1)
71       FORMAT(5F12.8 /)
         PRINT 72,VK,RAE
72       FORMAT(/ 2X,'K=',F12.8,5X,'RAE=',F12.8 ///)
111      CONTINUE
         STOP
         END
         FUNCTION P(X,T)
         P=T*(X**(T+1.0)-X**T)/2.0/(1.0-X**T)+.5
         RETURN
         END
         FUNCTION D(X,T)
         D=(T*T*X**T*(1.0-X)+T*X**T-T)/2.0/(X-X**(T+1.0))**2
         RETURN
         END
         FUNCTION F(Z,T,W)
         F=-.5*(1.0+T*(Z-1.0)/(Z-Z**(T+1.0)))-W
         RETURN
         END
         SUBROUTINE NEWTON(W,C,TT,Y)
         X0=.05
         I=1
10       P1=F(X0,C,W)
         F1=D(X0,C)
         XN=X0-P1/F1
         IF(XN.LE..0) GO TO 400
         DEL=ABS(XN-X0)/.0000002
         P2=F(XN,C,W)
         COF=ABS(P2)/.0000002
         IF(DEL.GT.1.0) GO TO 100
         IF(COF.GT.1.0) GO TO 100
         Y=XN
         GO TO 200
100      IF(I.GT.100) GO TO 400
         I=I+1
         X0=XN
         GO TO 10
400      PRINT 3,I
3        FORMAT(/ 2X,'I=',I3 /)
200      RETURN
         END
```

APPENDIX IV

PROGRAM FOR FINDING VALUES OF y_i^C OF THE RAYLEIGH DISTRIBUTION

```
C       TO FIND THE VALUES OF Y(I) FROM RAYLEIGH DIST.(SCALE)
        DIMENSION T(101),A(101),C(101),B(101)
        READ 1,K
1       FORMAT(I2)
        T(1)=.0
        A(1)=1.0
        C(1)=EXP(2.0)/2.0
        PRINT 3
3       FORMAT(/ 2X,' I',8X,' Y(I)' /)
        DO 101 I=1,K
        B(I)=T(I)+.000002
        CALL NEWRAN(B(I),C(I),Y)
        T(I+1)=Y/2.0/A(I)
        A(I+1)=1.0+T(I+1)*ALOG(T(I+1))/2.0/(1.0-T(I+1))
        Q=A(I+1)*2.0
        C(I+1)=EXP(Q)/Q
        IK=I+1
        PRINT 2,I,T(IK)
2       FORMAT(2X,I2,6X,F12.8)
101     CONTINUE
        STOP
        END
        SUBROUTINE NEWRAN(X0,C,XN)
        I=1
10      XN=X0-P(X0,C)/F1(X0)
        D=XN-X0
        DEL=ABS(D)
        DEL=DEL/.0000003
        COF=ABS(P(XN,C))/.00000002
        IF(DEL.GT.1.0) GO TO 100
        IF(COF.GT.1.0) GO TO 100
        Y=P(XN,C)
        GO TO 200
100     IF(I.GT.98) GO TO 400
        I=I+1
        X0=XN
        GO TO 10
400     PRINT 3,I
3       FORMAT(/ 2X,'I=',I2 /)
200     RETURN
        END
        FUNCTION P(Z,C)
        P=EXP(Z)/Z-C
        RETURN
        END
        FUNCTION F1(Y)
        F1=EXP(Y)*(Y-1.0)/Y**2
        RETURN
        END
```

APPENDIX V

PROGRAM FOR FINDING $\{\lambda_i^m\}_L$, CORRESPONDING a_i's, ARE (μ^*)'S OF THE ABLE μ^* BASED ON MULTIPLY CENSORED SAMPLES FROM THE LOGISTIC DISTRIBUTION

```
C       TO FIND THE OPTIMUM SPACING FOR MULTI-CENSORED LOGISTIC
C       SAMPLES (INCLUDING ANY KIND OF CENSORING CASES I.E.
C       SINGLE-CENSORED,DOUBLY-CENSORED,& MIDDLE-CENSORED )
        DIMENSION P(30,30),Y(100),S(100),T1(100),T2(100)
        DIMENSION T3(100),R(100)
        DIMENSION A(100)
        DIMENSION T4(100),T5(100),T6(100),R1(100),R2(100)
        DIMENSION R3(100),R4(100)
C       K=THE NO. OF ORDER STATISTICS SELECTED
C       J=THE NO. OF CENSORED INTERVALS
        READ(5,1) K,J
1       FORMAT(2I2)
C       P(I,1)=THE LEFT END POINT OF THE ITH. CENSORED-INTERVAL.
C       P(I,2)=THE RIGHT END POINT OF THE ITH. CENSORED-INTERVAL

        READ(5,2) ((P(I,L),L=1,2),I=1,J)
2       FORMAT(2F8.6)
        DK=K+1
        DO 30 I=1,K
        DI=I
30      R(I)=DI/DK
        CALL EXIST(R,K,J,P,LL)
        IF(LL.EQ.1) GO TO 60
        L=1
        DO 50 MU=1,J
        DO 50 NU=1,2
        L=L+1
50      Y(L)=P(MU,NU)
        M=2*J+1
        Y(1)=.0
        Y(M+1)=1.0
        CALL CHO1(M,K,J,Y,P,VK1,T1)
        IF(K.GT.1) GO TO 71
        DO 81 I=1,K
81      S(I)=T1(I)
        VK=VK1
        GO TO 20
71      CALL CHO2(M,K,J,Y,P,VK2,T2)
        CALL PMAX(T1,T2,K,VK,S)
        IF(K.EQ.2) GO TO 20
        CALL CHO3(M,K,J,Y,P,VK3,T3)
        CALL PMAX(S,T3,K,U1,R1)
        IF(K.GT.3) GO TO 74
        DO 82 I=1,K
82      S(I)=R1(I)
        VK=U1
        GO TO 20
74      CALL CHO4(M,K,J,Y,P,VK4,T4)
        CALL PMAX(R1,T4,K,U2,R2)
        IF(K.GT.4) GO TO 75
```

```
      DO 83 I=1,K
83    S(I)=R2(I)
      VK=U2
      GO TO 20
75    CALL CHO5(M,K,J,Y,P,VK5,T5)
      CALL PMAX(R2,T5,K,U3,R3)
      IF(K.GT.5) GO TO 76
      DO 84 I=1,K
84    S(I)=R3(I)
      VK=U3
      GO TO 20
76    CALL CHO6(M,K,J,Y,P,VK6,T6)
      CALL PMAX(R3,T6,K,U4,R4)
      DO 85 I=1,K
85    S(I)=R4(I)
      VK=U4
20    WRITE(6,5)
      A(1)=S(1)*(1.0-S(1))*S(2)/VK
      S(K+1)=1.0
      DO 55 I=2,K
55    A(I)=S(I)*(1.0-S(I))*(S(I+1)-S(I-1))/VK
5     FORMAT(/ 2X,23HTHE OPTIMUM SPACING IS: /)
      WRITE(6,6) (S(I),I=1,K)
6     FORMAT(2X,5F12.8)
      WRITE(6,8)
8     FORMAT(/ 2X,32HTHE COEFFICIENTS OF ESTIMATE ARE /)
      WRITE(6,6) (A(I),I=1,K)
      VK=3.0*VK
      WRITE(6,7) VK
7     FORMAT(/ 2X,18HASYMP. REL. EFF. =,F10.8 ///)
      GO TO 500
60    R(K+1)=1.0
      VK=R(1)*(1.0-R(1))**2
      DO 61 I=2,K+1
61    VK=VK+(R(I)-R(I-1))*(1.0-R(I)-R(I-1))**2
      WRITE(6,5)
      WRITE(6,6) (R(I),I=1,K)
      A(1)=R(1)*(1.0-R(1))*R(2)/VK
      R(K+1)=1.0
      DO 56 I=2,K
56    A(I)=R(I)*(1.0-R(I))*(R(I+1)-R(I-1))/VK
      VK=3.0*VK
      WRITE(6,8)
      WRITE(6,6) (A(I),I=1,K)
      WRITE(6,7) VK
500   STOP
      END
      SUBROUTINE EXIST(X,K,J,P,L1)
      DIMENSION X(100),P(30,30)
      DO 1 L=1,K
      IF((X(L).EQ.0.0).OR.(X(L).EQ.1.0)) GO TO 11
      DO 2 I=1,J
      IF((X(L).LT.P(I,2)).AND.(X(L).GT.P(I,1))) GO TO 11
2     CONTINUE
```

```
1       CONTINUE
        L1=1
        GO TO 12
11      L1=2
12      RETURN
        END
        SUBROUTINE PMAX(X,Y,K,Z,S)
        DIMENSION X(100),Y(100),S(100)
        VX=X(1)*(1.0-X(1))**2
        VY=Y(1)*(1.0-Y(1))**2
        X(K+1)=1.0
        Y(K+1)=1.0
        DO 1 I=2,K+1
        VX=VX+(X(I)-X(I-1))*(1.0-X(I)-X(I-1))**2
1       VY=VY+(Y(I)-Y(I-1))*(1.0-Y(I)-Y(I-1))**2
        IF(VX-VY) 2,3,3
2       Z=VY
        DO 11 I=1,K
11      S(I)=Y(I)
        GO TO 4
3       Z=VX
        DO 12 I=1,K
12      S(I)=X(I)
4       RETURN
        END
        SUBROUTINE CALOS(M1,P,Q,X)
        DIMENSION X(50)
        DM1=M1
        DO 1 I=1,M1
        DI=I
1       X(I)=((DM1-DI)*P+DI*Q)/DM1
        RETURN
        END
        SUBROUTINE CHOI(M,K,J,Y,P,VK,T)
        DIMENSION Y(100),P(30,30),S(100),T(100)
        DIMENSION TT(100),S2(100)
        Y(1)=.0
        Y(M+1)=1.0
        DO 1 I=1,K
1       T(I)=.0
        DO 2 I1=2,M
        DO 3 M1=1,K
        CALL CALOS(M1,Y(1),Y(I1),S)
        IF(K-M1.EQ.0) GO TO 10
        M2=K-M1+1
        CALL CALOS(M2,Y(I1),Y(M+1),S2)
        DO 4 I=1,M2-1
4       S(M1+I)=S2(I)
10      CALL EXIST(S,K,J,P,L)
        IF(L.EQ.2) GO TO 3
        DO 11 I=1,K
11      TT(I)=T(I)
        CALL PMAX(S,TT,K,VK,T)
3       CONTINUE
```

```
2       CONTINUE
        RETURN
        END
        SUBROUTINE CHO2(M,K,J,Y,P,VK,T)
        DIMENSION Y(100),P(30,30),S(100),T(100),TT(100)
        DIMENSION S1(100),S2(100)
        DO 1 I=1,K
1       T(I)=.0
        Y(1)=.0
        Y(M+1)=1.0
        DO 3 I1=2,M-1
        DO 3 I2=I1+1,M
        DO 4 M1=1,K-1
        CALL CALOS(M1,Y(1),Y(I1),S)
        DO 5 M2=1,K-M1
        CALL CALOS(M2,Y(I1),Y(I2),S1)
        DO 6 I=1,M2
6       S(M1+I)=S1(I)
        IF(M1+M2.EQ.K) GO TO 10
        M3=K-M1-M2+1
        CALL CALOS(M3,Y(I2),Y(M+1),S2)
        MM=M1+M2
        DO 7 I=1,M3-1
7       S(MM+I)=S2(I)
10      CALL EXIST(S,K,J,P,L)
        IF(L.EQ.2) GO TO 5
        DO 11 I=1,K
11      TT(I)=T(I)
        CALL PMAX(S,TT,K,VK,T)
5       CONTINUE
4       CONTINUE
3       CONTINUE
        RETURN
        END
        SUBROUTINE CHO3(M,K,J,Y,P,VK,T)
        DIMENSION Y(100),P(30,30),S(100),T(100)
        DIMENSION TT(100),S1(100),S2(100),S3(100)
        Y(1)=.0
        Y(M+1)=1.0
        DO 1 I=1,K
1       T(I)=.0
        DO 2 I1=2,M-2
        DO 2 I2=I1+1,M-1
        DO 2 I3=I2+1,M
        DO 3 M1=1,K-2
        CALL CALOS(M1,Y(1),Y(I1),S)
        DO 4 M2=1,K-M1-1
        CALL CALOS(M2,Y(I1),Y(I2),S1)
        DO 5 I=1,M2
5       S(M1+I)=S1(I)
        DO 6 M3=1,K-M1-M2
        CALL CALOS(M3,Y(I2),Y(I3),S2)
        DO 7 I=1,M3
7       S(M1+M2+I)=S2(I)
```

```
      IF(M1+M2+M3.EQ.K) GO TO 10
      M4=K-M1-M2-M3+1
      CALL CALOS(M4,Y(I3),Y(M+1),S3)
      MM=M1+M2+M3
      DO 8 I=1,M4-1
8     S(MM+I)=S3(I)
10    CALL EXIST(S,K,J,P,L)
      IF(L.EQ.2) GO TO 6
      DO 9 I=1,K
9     TT(I)=T(I)
      CALL PMAX(S,TT,K,VK,T)
6     CONTINUE
4     CONTINUE
3     CONTINUE
2     CONTINUE
      RETURN
      END
      SUBROUTINE CHO4(M,K,J,Y,P,VK,T)
      DIMENSION Y(100),P(30,30),S(100),T(100),TT(100)
      DIMENSION S1(100),S2(100),S3(100),S4(100)
      Y(1)=.0
      Y(M+1)=1.0
      DO 1 I=1,K
1     T(I)=.0
      DO 2 I1=2,M-3
      DO 2 I2=I1+1,M-2
      DO 2 I3=I2+1,M-1
      DO 2 I4=I3+1,M
      DO 11 M1=1,K-3
      CALL CALOS(M1,Y(1),Y(I1),S)
      DO 12 M2=1,K-M1-2
      CALL CALOS(M2,Y(I1),Y(I2),S1)
      DO 3 I=1,M2
3     S(M1+I)=S1(I)
      DO 13 M3=1,K-M1-M2-1
      CALL CALOS(M3,Y(I2),Y(I3),S2)
      DO 4 I=1,M3
4     S(M1+M2+I)=S2(I)
      MM=M1+M2+M3
      DO 14 M4=1,K-MM
      CALL CALOS(M4,Y(I3),Y(I4),S3)
      DO 5 I=1,M4
5     S(MM+I)=S3(I)
      IF(MM+M4.EQ.K) GO TO 10
      M5=K-MM-M4+1
      CALL CALOS(M5,Y(I4),Y(M+1),S4)
      DO 6 I=1,M5-1
6     S(MM+M4+I)=S4(I)
10    CALL EXIST(S,K,J,P,L)
      IF(L.EQ.2) GO TO 14
      DO 7 I=1,K
7     TT(I)=T(I)
```

```
      CALL PMAX(S,TT,K,VK,T)
14    CONTINUE
13    CONTINUE
12    CONTINUE
11    CONTINUE
2     CONTINUE
      RETURN
      END
      SUBROUTINE CHOS(M,K,J,Y,P,VK,T)
      DIMENSION Y(100),P(30,30),S(100),T(100),TT(100)
      DIMENSION S1(100),S2(100),S3(100),S4(100),S5(100)
      Y(1)=.0
      Y(M+1)=1.0
      DO 1 I=1,K
1     T(I)=.0
      DO 2 I1=2,M-4
      DO 2 I2=I1+1,M-3
      DO 2 I3=I2+1,M-2
      DO 2 I4=I3+1,M-1
      DO 2 I5=I4+1,M
      DO 11 M1=1,K-4
      CALL CALOS(M1,Y(1),Y(I1),S)
      DO 12 M2=1,K-M1-3
      CALL CALOS(M2,Y(I1),Y(I2),S1)
      DO 3 I=1,M2
3     S(M1+I)=S1(I)
      DO 13 M3=1,K-M1-M2-2
      CALL CALOS(M3,Y(I2),Y(I3),S2)
      DO 4 I=1,M3
4     S(M1+M2+I)=S2(I)
      MM=M1+M2+M3
      DO 14 M4=1,K-MM-1
      CALL CALOS(M4,Y(I3),Y(I4),S3)
      DO 5 I=1,M4
5     S(MM+I)=S3(I)
      DO 15 M5=1,K-MM-M4
      CALL CALOS(M5,Y(I4),Y(I5),S4)
      DO 6 I=1,M5
6     S(MM+M4+I)=S4(I)
      IF(MM+M4+M5.EQ.K) GO TO 10
      M6=K-MM-M4-M5+1
      CALL CALOS(M6,Y(I5),Y(M+1),S5)
      MO=MM+M4+M5
      DO 7 I=1,M6-1
7     S(MO+I)=S5(I)
10    CALL EXIST(S,K,J,P,L)
      IF(L.EQ.2) GO TO 15
      DO 8 I=1,K
8     TT(I)=T(I)
      CALL PMAX(S,TT,K,VK,T)
15    CONTINUE
14    CONTINUE
13    CONTINUE
12    CONTINUE
```

```
11      CONTINUE
2       CONTINUE
        RETURN
        END
        SUBROUTINE CHO6(M,K,J,Y,P,VK,T)
        DIMENSION Y(100),P(30,30),S(100),T(100),TT(100),S1(100)
        DIMENSION S2(100),S3(100),S4(100),S5(100),S6(100)
        Y(1)=.0
        Y(M+1)=1.0
        DO 1 I=1,K
1       T(I)=.0
        DO 2 I1=2,M-5
        DO 2 I2=I1+1,M-4
        DO 2 I3=I2+1,M-3
        DO 2 I4=I3+1,M-2
        DO 2 I5=I4+1,M-1
        DO 2 I6=I5+1,M
                DO 11 M1=1,K-5
        CALL CALOS(M1,Y(1),Y(I1),S)
        DO 12 M2=1,K-M1-4
        CALL CALOS(M2,Y(I1),Y(I2),S1)
        DO 3 I=1,M2
3       S(M1+I)=S1(I)
        DO 12 M3=1,K-M1-M2-3
        CALL CALOS(M3,Y(I2),Y(I3),S2)
        DO 4 I=1,M3
4       S(M1+M2+I)=S2(I)
        MM=M1+M2+M3
        DO 12 M4=1,K-MM-2
        CALL CALOS(M4,Y(I3),Y(I4),S3)
        DO 5 I=1,M4
5       S(MM+I)=S3(I)
        DO 12 M5=1,K-MM-M4-1
        CALL CALOS(M5,Y(I4),Y(I5),S4)
        DO 6 I=1,M5
6       S(MM+M4+I)=S4(I)
        MO=MM+M4+M5
        DO 13 M6=1,K-MO
        CALL CALOS(M6,Y(I5),Y(I6),S5)
        DO 7 I=1,M6
7       S(MO+I)=S5(I)
        IF(MO+M6.EQ.K) GO TO 10
        M7=K-MO-M6+1
        CALL CALOS(M7,Y(I6),Y(M+1),S6)
        DO 8 I=1,M7-1
8       S(MO+M6+I)=S6(I)
10      CALL EXIST(S,K,J,P,L)
        IF(L.EQ.2) GO TO 13
        DO 9 I=1,K
9       TT(I)=T(I)
        CALL PMAX(S,TT,K,VK,T)
13      CONTINUE
```

```
12      CONTINUE
11      CONTINUE
2       CONTINUE
        RETURN
        END
      EXAMPLE:
      KNOWN :   K=5, J=3
      KNOWN :   CENSORING INTERVALS( P(I,L) ):
      (0.135 , 0.274)  (0.4189 , 0.623)  (0.887 ,0.9456)
      DATA ARE READ IN AS FOLLOWS:
      1ST. CARD:0503
      2ND. CARD:0.13500 0.274
      3RD. CARD:0.41890 0.623
      4TH. CARD:0.88700 0.9456

      THE RESULT (OUTPUT) WILL BE PRINTED AS FOLLOWS:

 THE OPTIMUM SPACING IS:

  0.13500000  0.27695000  0.41890000  0.62300000  0.81150000

 THE COEFFICIENTS OF ESTIMATE ARE

  0.10003132  0.17584088  0.26054626  0.28520980  0.17837172

 ASYMP. REL. EFF. =0.96992131
```

REFERENCES

1. Aigner, D.J., & A.S. Goldberger, "Estimation of Pareto's Law from Grouped Observations," J. Amer. Statist. Assoc., 65, (1970), 712-723.

2. Balmer, D.W., M. Boulton, & R.A. Sack, "Asymptotic Solutions in the Estimation of the Location Parameter of the Cauchy Distribution," Preliminary Report, Department of Mathematics, University of Salford, England, (1972).

3. Berkson, J., "Tables for the Maximum Likelihood Estimate of the Logistic Function," Biometrics, 13, (1957), 28-34.

4. Chan, L.K., "Linear Quantile Estimates of the Location and Scale Parameters of the Logistic Distribution," Statist. Hefte, 10, (1969), 277-282.

5. __________, "Linear Estimation of the Location and Scale Parameters of the Cauchy Distribution Based on Sample Quantiles," J. Amer. Statist. Assoc., 65, (1970), 851-859.

6. __________, & S. Cheng, "Optimum Spacing for the Asymptotically Best Linear Estimate of the Location Parameter of the Logistic Distribution When Samples are Complete or Censored," Statist. Hefte 13, (1972), 41-58.

7. __________, & A.B.M.L. Kabir, "Optimum Quantiles for the Linear Estimation of the Parameters of the Extreme Value Distribution in Complete and Censored Samples," Naval Res. Logist. Quart., 16, (1969), 381-404.

8. __________, N.N. Chan, & E.R. Mead, "Best Linear Unbiased Estimates of the Parameters of the Logistic Distribution Based on Selected Order Statistics," J. Amer. Statist. Assoc., 66, (1971), 889-892.

9. Dyer, D.D., & C.W. Whisenand, "Best Linear Unbiased Estimation of the Parameter of the Rayleigh Distribution - Part I: Small Sample Theory for Censored Order Statistics," IEEE Trans. on Reliability, R-22, (1973), 27-34.

10. Eisenberger, I., "Test of Hypotheses and Estimation of the Correlation Coefficient Using Quantiles, I. II.," Technical Report No. 32-718, 32-755, JPL, Calif. Instit. Tech., Calif., (1965).

11. ______________, "Tests of Hypotheses and Estimation of the Correlation Coefficient Using Six and Eight Quantiles," Technical Report No. 32-1163, JPL, Calif. Instit. Tech., Calif. (1968).

12. ______________, "Testing the Mean and Standard Deviation of a Normal Distribution Using Quantiles," Technometrics, 10, (1968), 781-792.

13. Gupta, S.S., & M. Gnanadesikan, "Estimation of the Parameters of the Logistic Distribution," Biometrika, 53, (1966), 565-570.

14. ___________, A.S. Qureishi, & B.K. Shah, "Best Linear Unbiased Estimators of the Parameters of the Logistic Distribution Using Order Statistics," Technometrics, 9, (1967), 43-56.

15. Hassanein, K.M., "Analysis of Extreme-Value Data by Sample Quantiles for Very Large Samples," J. Amer. Statist. Assoc., 63, (1968), 877-888.

16. ______________, "Estimation of the Parameters of the Logistic Distribution by Sample Quantiles," Biometrika, 56, (1969), 684-687.

17. Harter, H.L., & A.H. Moore, "Maximum Likelihood Estimation from Censored Samples of the Parameters of a Logistic Distribution," J. Amer. Statist. Assoc., 62, (1967), 675-684.

18. Higuchi, I., "On the Solutions of Certain Simultaneous Equations in the Theory of Systematic Statistics," Ann. Instit. Statist. Math., 5, (1954), 77-90.

19. Johnson, N.L., & S. Kotz, Continuous Univariate Distributions - 1, Houghton Mifflin Co. Boston, (1970).

20. Kulldorff, G., "Maximum Likelihood Estimation of the Mean of a Normal Random Variable When the Sample is Grouped," Skand. Aktuarietidskr., 41, (1958), 1-17.

21. ____________, "Maximum Likelihood Estimation of the Standard Deviation of a Normal Random Variable When the Sample is Grouped," Skand. Aktuarietidskr., 41, (1958), 18-36.

22. ____________, "On the Asymptotic Optimum Spacings for the Estimation of the Scale Parameter of an Exponential Distribution Based on Sample Quantiles," Mimeographed Report, University of Lund, (1962).

23. ____________, "Estimation of One or Two Parameters of the Exponential Distribution on the Basis of Suitably Chosen Order Statistics," Ann. Math. Statist., 34, (1963), 1419-1431.

24. ____________, "On the Optimum Spacing of Sample Quantiles from a Normal Distribution, Part I., Part II.," Skand. Aktuarietidskr., 46, (1963), 143-161, 47, (1964), 71-87.

25. ______________, & K. Vännman, "Estimating Location and Scale Parameters of a Pareto Distribution by Linear Functions of Order Statistics," J. Amer. Statist. Assoc., 68, (1973), 218-227.

26. Likeš, J., "Minimum Variance Unbiased Estimates of the Parameters of Power-function and Pareto's Distribution," Statist. Hefte, 10, (1969), 104-110.

27. Malik, H.J., "Estimation of the Parameters of the Pareto Distribution," Metrika, 15, (1970), 126-132.

28. Moore, A.H. & H.L. Harter, "One-Order-Statistic Conditional Estimators of Shape Parameters of Limited and Pareto Distributions and Scale Parameters of Type II Asymptotic Distributions of Smallest and Largest Values," IEEE Trans. Reliability R-16, (1967), 100-103.

29. ______________________________, "Conditional Maximum-Likelihood Estimation, from Singly Censored Samples, of the Shape Parameters of Pareto and Limited Distributions," IEEE Trans. Reliability R-18, (1969), 76-78.

30. Mosteller, F., "On Some Useful Inefficient Statistics," Ann. Math. Statist., 17, (1946), 377-408.

31. Ogawa, J., "Contributions to the Theory of Systematic Statistics, I.," Osaka Math. J., 3, (1951), 175-213.

32. ________, "Determination of Optimum Spacings for the Estimation of the Scale Parameters of an Exponential Distribution Based on Sample Quantiles," Ann. Instit. Statist. Math., 12, (1960), 135-141.

33. ________, "Asymptotic Theory of Estimation of the Location and Scale Parameters Based on Selected Sample Quantiles," Preliminary Report, Department of Math., Statist., & Comp. Sc., The University of Calgary, Canada, (1972).

34. Quandt, R.E., "Old and New Methods of Estimation and the Pareto Distribution," Metrika, 10, (1966), 55-82.

35. Saleh, A.K.Md., & M.M. Ali, "Asymptotic Optimum Quantiles for the Estimation of the Parameters of the Negative Exponential Distribution," Ann. Math. Statist., 37, (1966), 143-151.

36. Tiku, M.L., "Estimating the Parameters of Normal and Logistic Distributions from Censored Samples," Austral. J. Statist., 10, (1968), 64-74.

37. Wang, Y.H., "Sequential Estimation of the Scale Parameter of the Pareto Distribution," Report 71-6, The Ohio State Univ., Ohio, (1971).